Week 1-2 Feedback

These notes provide feedback on some common issues from the first two weeks.

1 Circles vs Loops

The circle S^1 is not contractible, but a loop $f\colon I \to \mathbb{R}^2$ with basepoint x_0 is homotopic to a a point x_0 (or, more formally, the constant loop $I \mapsto x_0$). How can that be?

A map $f\colon X \to Y$ and its image $f(X) \subset Y$, considered as a topological space with the subspace topology, are completely different beasts. In fact, a loop can look much more complicated than a circle.

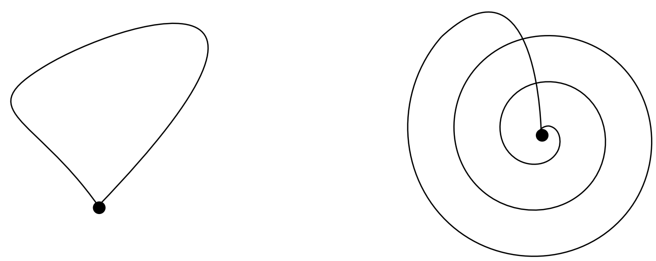


Figure 1: Homotopic loops whose images are highly unequivalent as topological spaces.

A homotopy between two loops $f, g\colon I \to \mathbb{R}^2$ moves one curve to another in $\mathbb{R}^2$, but apart from the base point x_0, all the loops f_t in the homotopy may as well have disjoint images. We need the ambient space $\mathbb{R}^2$ to make this homotopy happen.

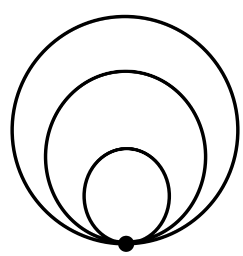


Figure 2: Homotopic loops whose images only intersect at the basepoint.

On the other hand, when talking about topological properties of S^1 we only have access to the space itself, and not any surrounding space. For example, a deformation retract of S^1 to a point x_0 would be a sequence of maps $f_t\colon S^1 \to S^1$ such that $f_0 = \mathrm{Id}_{S^1}$, $f_1 = \{x_0\}$ and $f_t(x_0) = x_0$. None of these can access points outside of S^1!

2 Quotient of a ball by its boundary

In this note we will show that

$$\mathbb{B}^n/S^{n-1} \cong S^n$$

with all the gory details. In topology one usually does not work in this level of detail, and once a certain intuition has developed, one can restrict to working with the diagrams, confident that the detailed derivation can be carried out if needed.

(1) The idea is to first consider a suitable map

$$f\colon \mathbb{B}^n \to S^n.$$

This map should be continuous and surjective (onto). In addition, we would like to construct the map such that

$$f(S^{n-1}) = \{x_0\}, \quad f(\mathbb{B}^n - S^{n-1}) = S^n - \{x_0\}$$

for some x_0 (for example, the "north pole" $x_0 = (0,\dots,0,1)$), and we want the restriction of f to $\mathbb{B}^n - S^{n-1}$ to be a bijection. Such a map, if it exists, automatically induces a bijection $\hat{f}\colon \mathbb{B}^n/S^{n-1} \to S^n$, such that $f = \hat{f} \circ q$, where $q\colon \mathbb{B}^n \to \mathbb{B}^n/S^{n-1}$ is the quotient map:

$$\begin{array}{ccc} \mathbb{B}^n & & \\ \downarrow q & \searrow f & \\ \mathbb{B}^n/S^{n-1} & \xrightarrow{\hat{f}} & S^n \end{array}$$

We only need to verify that this bijection is continuous, with continuous inverse.

To verify that $\hat{f}$ is continuous, let $V \subset S^n$ be open and $U = \hat{f}^{-1}(V)$. By the definition of the quotient topology, U is open in $\mathbb{B}^n/S^{n-1}$ if and only the preimage under the quotient map, $q^{-1}(U)$, is open in $\mathbb{B}^n$. But since $f = \hat{f} \circ q$, we have $q^{-1}(U) = f^{-1}(V)$. Since f is continuous, $f^{-1}(U)$ is open, and therefore also U. It follows that $\hat{f}$ is continuous. To show that this continuous bijection is a homeomorphism, we need to show that the inverse of $\hat{f}$ is continuous, or equivalently, that $\hat{f}$ is *open*: it maps open sets to open sets. While this can be shown directly in our case, we can also invoke the following general principle:

Lemma 2.1. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

(2) We have to show that a map f as described in Part (1) actually exists. We first define such a map, and then show how it can be arrived at. Note that there is no unique way of constructing this!

For $x \in \mathbb{B}^n$, set $r(x) = 1 - 2\sqrt{1 - \|x\|^2}$ and define a map $f\colon \mathbb{B}^n \to \mathbb{R}^{n+1}$ by

$$f(x) = \begin{cases} \left(\frac{x}{\|x\|}\sqrt{1 - r(x)^2}, r(x)\right) & x \neq 0 \\ (0,\dots,0,-1) & x = 0 \end{cases}$$

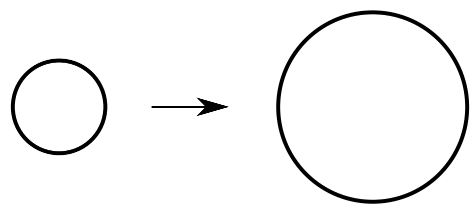
Note that:

1. For any $x \in \mathbb{B}^n$, $\|f(x)\|^2 = 1$ (that is, this is actually a map into S^n);
2. If $\|x\| = 1$ (that is, $x \in S^{n-1}$, the boundary of $\mathbb{B}^n$), then $f(x) = (0, \dots, 0, 1)$;
3. If $\|x\| < 1$, then $r(x) \in [-1, 1)$ and the map $x \mapsto f(x)$ gives rise to a bijection from $\mathbb{B}^n - S^{n-1}$ to $S^n - \{0, \dots, 0, 1\}$;
4. The map f is clearly continuous outside of $x = 0$. To see that it is continuous at 0, consider the limit of $f(x)$ as $\|x\| \to 0$. A direct calculation using the l'Hospital rule shows that the limit of the function $\sqrt{1 - r(x)^2}/\|x\|$ is indeed 0, which implies that the map is continuous.

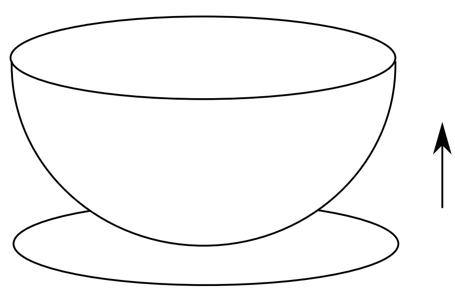
We have seen that the function f has all the properties required in Part (1). How does one arrive at such a function? The idea is to first construct the function visually in low dimension, and then figure out what it looks like in terms of equations. In our case, the construction consists of four steps, working with $\mathbb{B}^2$:

1. Construct a function $f_1 \colon \mathbb{B}^2 \to 2\mathbb{B}^2$ by simply scaling by a factor of 2:

$$f_1(x) = 2x.$$



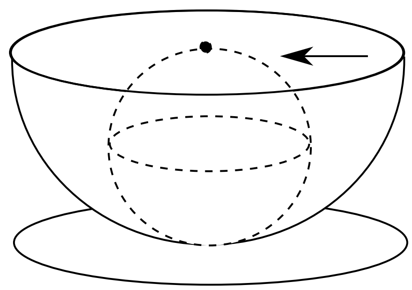
2. Add a new coordinate and map $2\mathbb{B}^2$ into $\mathbb{R}^3$ by taking every point to the point of a half-sphere of radius 2 above it:



One can figure out the relevant coordinates (map every x to the lower point on a 2-sphere with whose first two coordinates are given by x, and then translate the centre):

$$f_2(x) = (x, 2 - \sqrt{4 - \|x\|^2}).$$

3. Place a 2-sphere with center $(0,0,1)$ and radius 1 inside the "cup", and map every point on the cup onto the surface of this sphere. The resulting map is simply the map projecting a point in $\mathbb{R}^3$ horizontally onto a sphere centred at $(0,0,1)$. In particular, one sees that the boundary of the cup lands on the north pole, while every other point has a unique corresponding point on the sphere.



Working this out in coordinates, the x_3 coordinate does not change, but the first two coordinates have to be scaled until they hit the sphere. The projection onto the first two coordinates of a point on the sphere centred at $(0,0,1)$ has length $\sqrt{1-(x_3-1)^2}$. This means that one has to scale the coordinates (x_1,x_2) of every point by a factor of $\sqrt{1-(x_3-1)^2}/\sqrt{x_1^2+x_2^2}$ to get a point on the middle sphere. Of course, one has to treat the case $(x_1,x_2)=(0,0)$ separately. The projection map for $x \neq 0$ is given by

$$f_3(x) = \left(\frac{\sqrt{1-(x_3-1)^2}}{\sqrt{x_1^2+x_2^2}} x_1, \frac{\sqrt{1-(x_3-1)^2}}{\sqrt{x_1^2+x_2^2}} x_2, x_3 \right).$$

4. In the last step, we simply translate the sphere centered at $(0,0,1)$ to make it centred at the origin, that is, $f_4(x_1,x_2,x_3) = (x_1,x_2,x_3-1)$.

The resulting map $\mathbb{B}^2 \to S^2$ is given by the composition $f = f_4 \circ f_3 \circ f_2 \circ f_1$, which looks precisely as above (because in the third step we divide by something that could be 0, we have to make the case distinction at the origin). Once this map has been constructed in three dimension, we see that we can extend it to higher dimension without any difficulties.

3 Order of path composition

In the first version of the lecture notes for Lecture 4, the order of composition was wrong: $f' * f$ means the composition of f' and f, with f' coming before f (i.e., $f'(1) = f(0)$.

4 Continuity of compositions

It has been pointed out that the fact that the composition of two paths is continuous is not obvious. This follows from the **Pasting Lemma**, mentioned in Lecture 3.

Lemma 4.1. *(Pasting Lemma) Let $X = A \cup B$, with A, B both closed subspaces of a topological space X. Let $f\colon X \to Y$ be a function and assume that $f|_A$ and $f|_B$ are continuous. Then f is continuous.*

The proof is easy and can be found, for example, in Wikipedia. This result is used whenever we construct a map on a topological space based on maps on subspaces (for example, when compositing homotopies). We will not always explicitly refer to the Pasting Lemma.