Week 3-4 Feedback

These notes provide feedback on some common questions and issues arising in weeks 3-4.

1 Homotopies in path-connected spaces

It was mentioned in the lectures that any two paths in a path connected space are homotopic. The most natural way one might attempt to show this, namely by giving a direct homotopy from one path to the other, does not work. Instead, one can use the fact that each path is homotopic to its starting point: if $f\colon I \to X$ with $f(0) = x_0$, then $f_t(s) = f((1-t)s)$ is a homotopy of f to the constant path $e_0(s) = x_0$, and similarly any path g with $g(0) = x_1$ is homotopic to the constant path at x_1, e_1. Since X is path-connected, for any $x_0, x_1 \in X$ there is a path connecting x_0 to x_1, and therefore a homotopy of the constant path at x_0 to the constant path at x_1, $e_0 \simeq e_1$. By the properties of the homotopy relation, $f \simeq e_0 \simeq e_1 \simeq g$.

2 Product topology

Given the product of two topological spaces $X \times Y$, the product topology has products $U \times V$ of open sets in X and Y as *basis*, but not every open set looks like this.

3 Surjectivity of covering

In the definition of a covering $p\colon \tilde{X} \to X$ we did not explicitly state that p has to be surjective. By a slight modification of the definition of a cover, we can ensure that p is surjective: namely, if we require that in the covering $X = \bigcup_\alpha U_\alpha$, the preimage $p^{-1}(U_\alpha)$ is not empty. That p is surjective then follows from the fact that $p^{-1}(U_\alpha) = \bigsqcup_\beta V_\alpha^\beta$, with $p|_{V_\alpha^\beta}\colon V_\alpha^\beta \to U_\alpha$ a homeomorphism. In particular, this means that every $x \in X$ lies in some U_α that is homeomorphic to *a part of* $\tilde{X}$, and hence has a preimage under p.

While our definition does not exclude the case that some $p^{-1}(U_\alpha)$ might be empty, in practice this will not be too much of an issue. For example, for the homotopy lifting property we require an "initial lift" $g\colon Y \times \{0\} \to \tilde{X}$ with $p \circ f_0$, and this requirement ensures that the relevant part of X is "under" $\tilde{X}$.

4 How to derive homotopies I

Let $X \subset \mathbb{R}^4$ by the union of the xy-plane and the zw-plane, and consider the space $\mathbb{R}^4 - X$. This space is homotopy equivalent to the torus $\mathbb{T}^2$. How do we go about showing this? Since we can't really visualize $\mathbb{R}^4$, we have to work with coordinates. For this particular problem, it helps to think of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, so that the xy-plane is described by $X_1 = \mathbb{R}^2 \times \{(0,0)\}$ and the zw-plane by $X_2 = \{(0,0)\} \times \mathbb{R}^2$. Therefore,

$$\mathbb{R}^4 - X_1 = \{(x,y,z,w) \in \mathbb{R}^4 \colon (z,w) \neq (0,0)\} = \mathbb{R}^2 \times (\mathbb{R}^2 - \{(0,0)\})$$
$$\mathbb{R}^4 - X_2 = \{(x,y,z,w) \in \mathbb{R}^4 \colon (x,y) \neq (0,0)\} = (\mathbb{R}^2 - \{(0,0)\}) \times \mathbb{R}^2.$$

Now $\mathbb{R}^2 - \{(0,0)\} \simeq S^1$ by the homotopy $f_t(x) = (1-t)x + tx/\|x\|$. Therefore, $\mathbb{R}^4 - X_1 \simeq \mathbb{R}^2 \times S^1$ and $\mathbb{R}^4 - X_2 \simeq S^1 \times \mathbb{R}^2$ by the homotopy that does not do anything on $\mathbb{R}^2$ and moves $\mathbb{R}^2 - \{(0,0)\}$ to S^1.

$$\mathbb{R}^4 - X = \mathbb{R}^4 - (X_1 \cup X_2) = (\mathbb{R}^4 - X_1) \cap (\mathbb{R}^4 - X_2) = S^1 \times S^1,$$

which is the torus.

5 How to derive homotopies II

Probably the most important way to derive homotopies or that spaces are homotopy equivalent is by using the properties of homotopy equivalence (symmetry, transitivity), and sometimes introduce auxiliary spaces. For example, to show that $X \simeq Z$, a common strategy is to find a space Y such that $X \simeq Y$ and $Z \simeq Y$, rather than trying to produce explicit maps from X to Z. A typical example is Problem (2.7):

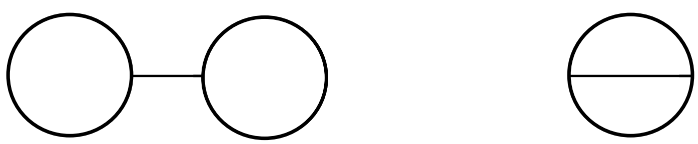

Figure 1: Left: the eyeglasses graph. Right: the theta graph.

The trick here is to note that both of these spaces are homotopy equivalent to the "figure eight" ∞.

6 On graphs and visualisation

Topology is a visual subject, and sometimes one is required to make sketches or drawings to visualize a particular relationship. Obviously, these do not constitute a proof and there is not unique way of doing this. What's important here is that the *idea* of a concept or of a construction comes across clearly. Interestingly, the visual aspect of topology somewhat carries over to the algebraic and abstract theory, which relies heavily on diagrams of maps and functions (points and morphisms). See also the film "It's my turn" (1980), https://www.youtube.com/watch?v=etbcKWEKnvg.