# Lecture 1

*"[...] so far as geometry is concerned, we need still another analysis which is distinctly geometrical or linear and which will express situation [situs] directly as algebra expresses magnitude directly. "*

— G.W.Leibniz, Letter to Huygens, September 8, 1679

Topology is the study of properties of spaces that are invariant under continuous deformations; it is concerned with concepts such as "nearness", "neighbourhood", and "convergence". An often cited example is that a cup is topologically equivalent to a torus, but not to a sphere. But what exactly does "topologically equivalent" mean?

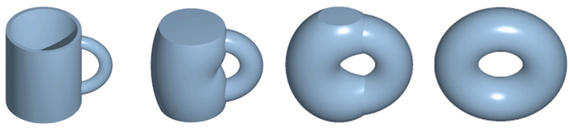

Figure 1.1: A cup morphing into a torus. (c) LucasVB (Wikipedia)

The roots of topology go back to the work of Leibniz and Euler in the 17th and 18th century. It was only towards the end of the 19th century, through the work of Poincaré, that topology began taking shape as a subject in its own right. His seminal paper "Analysis Situs" from 1895 introduced, among other things, the idea of a homeomorphism and the fundamental group. Nowadays, topological ideas are an indispensable part of many fields of mathematics, ranging from number theory to partial differential equations.

## 1.1 Background and terminology

This course assumes familiarity with metric spaces, linear algebra, some algebra, and calculus. We use capital letters $X, Y, Z$ to denote sets and $A, B, C$ to denote subsets. The notation $A \subset X$ denotes (not necessarily proper) inclusion, and $X - A$ is the complement of $A$ in $X$. Following common pedantry, we will refer to "sets" of sets as collections of sets, to avoid logical catastrophes (the "set of sets that are not members of themselves"). Other notation will be explained as it arises. Following common convention, we use arrows and diagrams to describe maps between sets. A diagram such as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

describes three sets $X, Y, Z$ and three functions, $f\colon X \to Y$, $g\colon Y \to Z$, and $h\colon X \to Z$. Such a diagram is *commutative* if all compositions agree; here, this means that $h = g \circ f$. We sometimes use the notation $X \hookrightarrow Y$ to denote an *injective* (or one-to-one) map (for example, the map $x \mapsto x$ arising from an inclusion $X \subset Y$), and $X \twoheadrightarrow Y$ for a *surjective* (or onto) map. We recall the definition of a metric space.

**Definition 1.1.** A *metric space* is a set $X$, together with a function $\mathrm{dist}\colon X \times X \to \mathbb{R}$, such that

1. (positivity) $\mathrm{dist}(x, y) \geq 0$ for all $x, y \in X$, with equality if and only if $x = y$;

2. (symmetry) $\mathrm{dist}(x, y) = \mathrm{dist}(y, x)$;

3. (triangle inequality) $\mathrm{dist}(x, z) \leq \mathrm{dist}(x, y) + \mathrm{dist}(y, x)$.

Well known examples include $\mathbb{R}^n$ with the Euclidean distance or any distance induced by a norm, or the space $C([0, 1])$ of real-valued, continuous functions on the interval $[0, 1]$, with the metric

$$\mathrm{dist}(f, g) = \int_0^1 |f(x) - g(x)| \, \mathrm{d}x.$$

Given a metric space $(X, \mathrm{dist})$ and $x_0 \in X$, we denote by

$$B(x_0, \varepsilon) = \{x \in X\colon \ \mathrm{dist}(x, x_0) < \varepsilon\}$$

the open ball of radius $\varepsilon$ centred on $x_0$.

**Definition 1.2.** Let $(X, d)$ be a metric space. A set $U \subset X$ is called *open* in $X$, if for every $x \in U$ there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. A subset of $X$ is *closed* in $X$ if its complement is open.

Clearly, the empty set $\emptyset$ and the whole space $X$ are open. Moreover, the union of *any* collection of open sets is open, and the intersection of a *finite* collection of open sets is open (show this!). It turns out that these properties allows us to define open sets and neighbourhoods beyond metric spaces.

## 1.2 Topological spaces

**Definition 1.3.** A **topological space** is a set $X$, together with a collection $\Omega$ of subsets of $X$, such that

(i) $\emptyset \in \Omega$ and $X \in \Omega$;

(ii) if $\{U_i\}_{i\in I} \subset \Omega$, then $\bigcup_{i\in I} U_i \in \Omega$;

(iii) if $U, V \in \Omega$, then $U \cap V \in \Omega$.

The sets in $\Omega$ are called **open sets** and their complements in $X$ are called **closed sets**.

Note that point (iii) implies that any *finite* intersection of open sets is again open.

**Definition 1.4.** Let $(X, \Omega)$ be a topological space. A **neighbourhood** of a point $x \in X$ is a set $N$ such that there exists $U \in \Omega$ with $x \in U \subset N$.

While formally a topological space consists of the pair $(X, \Omega)$, we often omit the mention of $\Omega$. Unless otherwise stated, when considering a metric space $(X, \mathrm{dist})$ we will always use the *metric topology*, i.e., the topology whose open sets are given by Definition 1.2.

**Example 1.5.** Different metric spaces can give rise to the same topology. In fact, any norms on a finite-dimensional vector space give rise to the same topology. Consider, for example, $X = \mathbb{R}^n$ with the norms $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|x\|_\infty = \max_i |x_i|$, and the corresponding distance functions $d_1(x, y) = \|x - y\|_1$ and $d_\infty(x, y) = \|x - y\|_\infty$. The norm inequalities

$$\|x\|_\infty \leq \|x\|_1 \leq n \cdot \|x\|_\infty$$

ensure that for any set $U \subset X$ and $x_0 \in U$, there is an open ball around $x_0$ in $U$ with respect to one of these norms, if and only if there is one with respect to the other.

Specifying a topology is not always easy. Just like one can specify a vector space by giving a basis, one can also describe a topology in terms of a basis.

**Definition 1.6.** Let $(X, \Omega)$ be a topological space. A collection $\mathcal{B} \subset \Omega$ is called a **basis** for the topology $\Omega$, if for all $U \in \Omega$ there exists a collection $\{B_i\}_{i\in I} \subset \mathcal{B}$ such that $\bigcup_{i\in I} B_i = U$. Given $x \in X$, a collection $\mathcal{B}$ is called a **neighbourhood basis** for $x$, if for every open set $U \in \Omega$ with $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

**Exercise 1.7.** [1] Show that every metric space $(X, d)$ is *first countable*: every point in $X$ has a *countable* neighbourhood basis. Next, show that $\mathbb{R}$ with the *cofinite topology*, i.e., the topology whose open sets are the complements of finite sets, is not first countable. Hence, conclude that there are topological spaces that do not arise from a metric.

[1] The exercises in these notes are not examinable, they are only there for your enjoyment (though you may benefit intellectually from attempting them, consider them as "cross training").

**Example 1.8.** The open intervals $(a, b)$ form a basis of the metric topology on $\mathbb{R}$.

Given a collection of subsets $\mathcal{B}$ of a topological space $(X, \Omega)$, we say that $\mathcal{B}$ **generates** the topology if $\mathcal{B}$ is a basis of $\Omega$. Any collection of subsets $\mathcal{B}$ that is closed under finite intersections generates a topology, whose open sets are just the unions of elements of $\mathcal{B}$.

## Product spaces

**Definition 1.9.** Let $X, Y$ be topological spaces. The **product topology** on $X \times Y$ is the topology generated by sets of the form $U \times V$, with $U \subset X$ open and $V \subset Y$ open.

Thus every open set in the product topology can be written as a (generally infinite) union of "rectangles" $U \times V$.

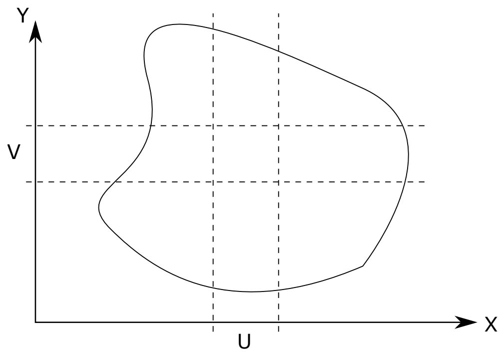


Figure 1.2: The product topology

**Exercise 1.10.** One can define the product topology on $\mathbb{R}^n$ recursively by setting $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ and $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ for $n \geq 2$. Show that the product topology on $\mathbb{R}^n$ is the same as the metric topology. (One can interpret the first as the topology generated by "open boxes", and the second as the topology generated by "open balls".)

## Subspaces

**Definition 1.11.** Let $(X, \Omega)$ be a topological space and $A \subset X$ a subset. The **subspace topology** on $A$ consists of the open sets

$$\Omega|_A = \{U \cap A \mid U \in \Omega\}.$$

**Example 1.12.** The closed interval $[0, 1] \subset \mathbb{R}$. Note that $(1/2, 1]$ is open in the subspace topology on $[0, 1]$!

**Example 1.13.** The **unit sphere**,

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n} x_i^2 = 1\}.$$

Note that the superscript denotes the dimension of the sphere, and not that of the ambient space in which the sphere lives.

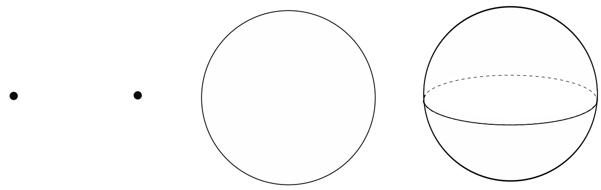

Figure 1.3: The spheres $S^0$, $S^1$ and $S^2$.

**Example 1.14.** The **unit ball**

$$\mathbb{B}^n = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 \leq 1\}.$$

**Example 1.15.** The **topological torus**, defined as product of 1-spheres (circles)

$$\mathbb{T}^1 = S^1, \quad \mathbb{T}^n = \mathbb{T}^{n-1} \times \mathbb{T}^1 = S^1 \times \cdots \times S^1 \ (n \text{ times}),$$

for $n \geq 2$. To justify the terminology "torus", consider the parametrization of a torus $X$ in $\mathbb{R}^3$ as the set of $(x, y, z)$ such that

$$\begin{aligned} x(\theta, \varphi) &= (a\cos(\theta) + b)\sin(\varphi) \\ y(\theta, \varphi) &= (a\cos(\theta) + b)\cos(\varphi) \\ z(\theta, \varphi) &= a\sin(\theta). \end{aligned}$$

for $\theta, \varphi \in [0, 2\pi)$ and fixed $0 < a < b$.

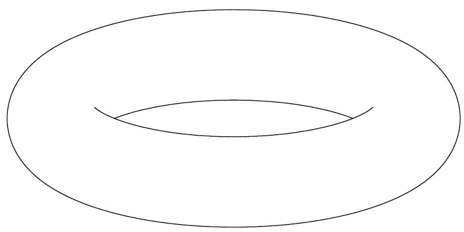

Figure 1.4: The embedded torus. The large circle going through the torus has radius $b$ and the small circle bounding a section has radius $a$.

The product of spheres, $\mathbb{T}^2 = S^1 \times S^1$, can in turn be parametrized as the set of $(x_1, y_1, x_2, y_2) \in \mathbb{R}^4$ such that

$$x_1 = \cos(\theta),\ y_1 = \sin(\theta),\ x_2 = \cos(\varphi),\ y_2 = \sin(\varphi),$$

for $\theta, \varphi \in [0, 2\pi)$. This gives rise to a function

$$f\colon S^1 \times S^1 \to X$$
$$\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} (ax_1 + b)y_2 \\ (ax_1 + b)x_2 \\ ay_1 \end{pmatrix}$$

As we will see, this map is a continuous bijection with continuous inverse, also called a homeomorphism.

## 1.3 Homeomorphism

**Definition 1.16.** Let $X, Y$ be topological spaces. A function $f\colon X \to Y$ is called **continuous**, if for any open set $V \subset Y$, the preimage $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open in $X$.

We will refer to a continuous function simply as a **map**. Unless stated otherwise, all functions in this lecture are continuous.

**Example 1.17.** The identity map, $\mathrm{Id}_X : X \to X$, $x \mapsto x$, is clearly continuous, as is the inclusion $\iota\colon A \hookrightarrow X$ of a subset $A \subset X$ with the subspace topology.

**Example 1.18.** The map $\mathbb{R} \to S^1$ given by $t \mapsto (\cos(t), \sin(t))$ is continuous.

**Example 1.19.** The map $\mathbb{R} \to \mathbb{R}$, given by

$$x \mapsto \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not continuous.

**Definition 1.20.** Let $X, Y$ be topological spaces. A map $f\colon X \to Y$ is called a **homeomorphism**, if there exists a map $g\colon Y \to X$ such that

$$f \circ g = \mathrm{Id}_Y, \quad g \circ f = \mathrm{Id}_X.$$

If a homeomorphism between $X$ and $Y$ exists, these spaces are called **homeomorphic**, written $X \cong Y$.

**Example 1.21.** The identity $\mathrm{Id}_X$ is clearly a homeomorphism. The map $\mathbb{R} \to \mathbb{R}$, $x \to x^3$ is a homeomorphism, while $x \mapsto x^2$ is not (it is not invertible).

When we speak of spaces being “topologically equivalent”, we mean that they are homeomorphic. Topology does not distinguish between homeomorphic spaces.

**Exercise 1.22.** Show that the map $f\colon \mathbb{T}^2 \to X$ from Example 1.15 is a homeomorphism. This requires figuring out the inverse of this map and showing that both the map, and its inverse, are continuous.

## 1.4 The Fundamental Problem

The Fundamental Problem in topology is to classify spaces up to homeomorphism. More precisely, we would like to have a way of answering the question:

Given topological spaces $X, Y$, is $X \cong Y$?

**Example 1.23.** $\mathbb{R}^0 = \{\mathrm{pt}\}$ (a single point) is not homeomorphic to $\mathbb{R}^1$ (a line).

How about $\mathbb{R}^1$ and $\mathbb{R}^2$? One might think that they are topologically not the same, as one is "somehow bigger". If they were homeomorphic, one could fine a continuous and continuously invertible parametrization of the plane by a line. It turns out that the problem of showing that two real vector spaces of different dimension are not homeomorphic is not trivial. The tools developed in this module will allow to prove the following.

**Theorem 1.24.** *(Invariance of Domain, Brouwer 1910) $\mathbb{R}^m \cong \mathbb{R}^n$ if and only if $m = n$.*

**Exercise 1.25.** Try to show that $S^2 \not\cong \mathbb{T}^2$.

To show that two spaces are homeomorphic, one only needs to provide a homeomorphism. To show that they are *not* homeomorphic is more difficult, and amounts to finding a property that is a) invariant under homeomorphism, and b) is satisfied by one of the spaces but not the other. As we will see, algebraic invariants such as the fundamental group (the main topic of this module) allow to accomplish this.