# Lecture 10

The proof that $\pi_1(S^1, 1)$ relied on the fact that coverings satisfy the Homotopy Lifting Property. In this lecture we prove this fact.

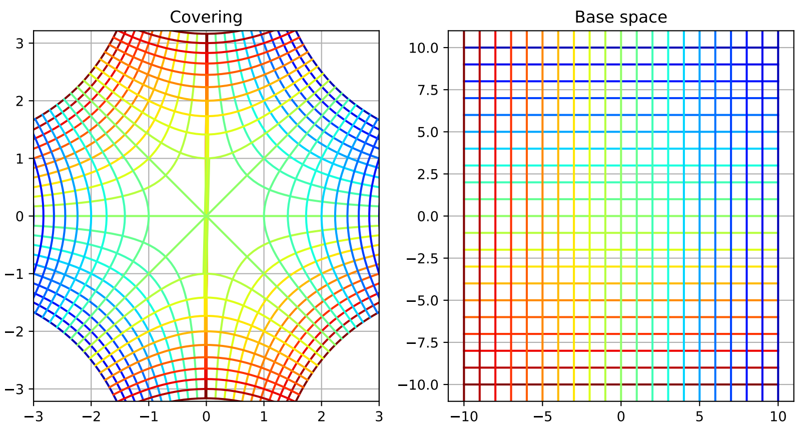


Figure 10.1: The covering map $\mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$, $z \mapsto z^2$ with the liftings of two paths (a vertical and a horizontal line).

## 10.1 Homotopy lifting for coverings

**Proposition 10.1.** *Covering maps satisfy the homotopy lifting property.*

The proof follows at once from the following lemma.

**Lemma 10.2.** *Let $p\colon \tilde{X} \to X$ be a covering and let $F\colon Y \times I \to X$ be a homotopy. Let $g\colon Y \times \{0\} \to \tilde{X}$ be such that $p \circ g = f_0$. Then for every $y_0 \in Y$ there exists an open set $N$ with $y_0 \in N \subset Y$ and a* unique *homotopy (depending on $N$)*

$$\tilde{F}_N\colon N \times I \to \tilde{X}$$

*such that* $p \circ \tilde{F}_N = F|_{N\times I}$ *and* $\tilde{f}|_{N0} = g|_N$. *Moreover, if* $y_0 \in M \subset Y$ *is another such open neighbourhood, then*

$$\tilde{F}_M|_{(M\cap N)\times I} = \tilde{F}_N|_{(M\cap N)\times I} = \tilde{F}_{M\cap N}. \tag{10.1}$$

*Proof of Proposition 10.1.* Cover $Y \times I$ with open sets $N \times I$ as guaranteed by Lemma 10.2. We thus get a family of homotopy lifts $F_N \colon N \times I \to \tilde{X}$. By (10.1), these maps coincide on their intersection, and thus give a homotopy $F \colon Y \times I \to \tilde{X}$ with the desired properties. By the Pasting Lemma, the resulting function $F$ is continuous, concluding the proof. □