Lecture 11

In this lecture we will complete the last missing piece in the derivation of the fundamental group $\pi_1(S^1, 1)$.

11.1 The local Homotopy Lifting Property

Lemma 11.1. *Let $p\colon \tilde{X} \to X$ be a covering and let $F\colon Y \times I \to X$ be a homotopy. Let $g\colon Y \times \{0\} \to \tilde{X}$ be such that $p \circ g = f_0$. Then for every $y_0 \in Y$ there exists an open set N with $y_0 \in N \subset Y$ and a* unique *homotopy (depending on N)*

$$\tilde{F}_N\colon N \times I \to \tilde{X}$$

such that $p \circ \tilde{F}_N = F|_{N\times I}$ and $(\tilde{f}_N)_0 = g|_{N\times\{0\}}$. Moreover, if M is another such neighbourhood, with $y_0 \in M \subset Y$, then

$$\tilde{F}_M|_{(M\cap N)\times I} = \tilde{F}_N|_{(M\cap N)\times I} = \tilde{F}_{M\cap N}. \tag{11.1}$$

Proof. Let $p\colon \tilde{X} \to X$ be a covering map. Assume we have a homotopy $F\colon Y\times I \to X$ and "initial data" $g\colon Y \to \tilde{X}$, so that $p \circ g = f_0$. By the definition of a covering, we have an open cover $\{U_\alpha\}$ of X, and for each α a collection of disjoint subsets $\{V_\alpha^\beta\}$ of $\tilde{X}$ such that $p^{-1}(U_\alpha) = \bigsqcup V_\alpha^\beta$, and the restriction $p|_{V_\alpha^\beta}\colon V_\alpha^\beta \to U_\alpha$ is a homeomorphism. For every pair (α, β), denote by $q_{\alpha,\beta}\colon U_\alpha \to V_\alpha^\beta$ the inverse of this homeomorphism. Since F is continuous, for every $(y,t) \in Y \times I$ there exists an open neighbourhood $N \times (a,b) \subset Y \times I$ and an index α such that $F(y,t) \in U_\alpha$ for $(y,t) \in N \times (a,b)$. For every fixed y and as t ranges over I, we get various subsets with this property, and since I is compact, there are finitely many such $N_i \times (a_i, b_i)$ covering $\{y\} \times I$. Set $N = \cap_i N_i$ and choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for every i there is an α with $F(N \times (t_i, t_{i+1})) \subset U_\alpha$.

We now claim that there is a sequence of maps $\tilde{F}_N^k$ such that

1. $\tilde{F}_N^k\colon N \times [0, t_k] \to \tilde{X}$ is a lift of $F|_{N\times[0,t_k]}$;
2. $(\tilde{F}_N^k)_0 = g|_N$;
3. $\tilde{F}_N^{k+1}|_{N\times[0,t_k]} = \tilde{F}_N^k$.

Moreover, these three properties determine the sequence $\{\tilde{F}_N^k\}$ uniquely. We then set $F_N = F_N^n$. We construct the sequence of maps $\tilde{F}_N^k$ by induction. Clearly, there is only one way to define $\tilde{F}_N^0$ on $N \times [0,0]$ such that $(\tilde{F}_N^0)_0 = g|_N$. Assume now that we have a sequence of maps $\tilde{F}_N^j$ up to $j = k$. By assumption, there is an index α such that $F(N \times [t_k, t_{k+1}]) \subset U_\alpha$. By making N smaller, if necessary, we can assume that $\tilde{F}_N^k|_{N\times\{t_k\}} \subset V_\alpha^\beta$ for some β, and that if we define

$$\tilde{E} = q_{\alpha,\beta} \circ F|_{N\times[t_k,t_{k+1}]},$$

then

$$\tilde{E}|_{N\times\{t_k\}} = \tilde{F}_N^k|_{N\times\{t_k\}}.$$

Now define the extension

$$\tilde{F}_N^{k+1}(z,t) = \begin{cases} \tilde{F}_N^k(z,t), & t \le t_k \\ \tilde{E}(z,t) & t \in [t_k, t_{k+1}]. \end{cases}$$

By the Pasting Lemma, $\tilde{F}_N^{k+1}$ is continuous. By construction, the resulting map satisfies conditions (1)-(3) above.

Assume now that we have two maps, $\tilde{F}_N$, $\tilde{F}'_N$, constructed in this fashion. It is enough to show that, for any $z \in N$, $\tilde{F}_N|_{\{z\}\times I} = \tilde{F}'_N|_{\{z\}\times I}$. As before, let $0 = t_0 < t_1 < \cdots < t_m = 1$ be a partition such that $F(\{z\} \times [t_j, t_{j+1}]) \subset U_\alpha$. We proceed by induction. It is clear that both maps have to coincide on $\{z\} \times [0,0]$, as both have to match $g(z,0)$ there. Assume that $\tilde{F}'_N = \tilde{F}_N$ on $[0, t_k]$. Since $[t_k, t_{k+1}]$ is connected, there exists a unique β such that $\tilde{F}_N(\{z\} \times [t_k, t_{k+1}])$ is contained in V_α^β. Similarly, there is a unique β' such that $\tilde{F}'_N(\{z\} \times [t_k, t_{k+1}])$ is in $V_\alpha^{\beta'}$. But since $\tilde{F}_N(z, t_k) = \tilde{F}'_N(z, t_k)$, we have to have $\beta = \beta'$. By construction of the extension $\tilde{E}$, the two maps also coincide on $\{z\} \times [0, t_{k+1}]$.

The proof also shows that if we take two neighbourhoods N, M with the properties just derived, then by uniqueness we have $\tilde{F}_M|_{(M\cap N)\times I} = \tilde{F}_N|_{(M\cap N)\times I}$. □