Lecture 12

We computed the fundamental group of some elementary spaces, but haven't really seen what this means yet. For example, if we denote the closed disk in $\mathbb{C} \cong \mathbb{R}^2$ by

$$\mathbb{D}^2 := \{z \in \mathbb{C} \mid |z| \leq 1\},$$

then

$$\pi_1(\mathbb{D}^2, 1) = \{0\}, \quad \pi_1(S^1, 1) \cong \mathbb{Z}.$$

What does this say about the underlying topological spaces? As we will see, this implies (for example) that S^1 cannot be a retract of $\mathbb{D}^2$, which in turn has other consequences such as the Brouwer Fixed Point Theorem (a map $\mathbb{D}^2 \to \mathbb{D}^2$ has a fixed-point).

12.1 Induced homomorphisms

Recall that a pair of spaces is a pair of topological spaces (X, A) with $A \subset X$.

Definition 12.1. A **map of pairs**

$$f\colon (X, A) \to (Y, B)$$

is a map $f\colon X \to Y$ such that $f(A) \subset B$.

Example 12.2. The typical example is when $A = \{x_0\}$ and $B = \{y_0\}$, in which case we write $f\colon (X, x_0) \to (Y, y_0)$ to denote a map with $f(x_0) = y_0$.

Example 12.3. Consider the two-fold cover $f\colon (S^1, 1) \to (S^1, 1)$, $z \mapsto z^2$.

Definition 12.4. The **induced homomorphism** of $f\colon (X, x_0) \to (Y, y_0)$ is the map

$$\begin{aligned} f_*\colon \pi_1(X, x_0) &\to \pi_1(Y, y_0) \\ [\alpha] &\mapsto [f \circ \alpha]. \end{aligned}$$

The induced homomorphism is also sometimes called a *push-forward*.

Lemma 12.5. *The map f_* is a group homomorphism.*

Proof. We first have to verify that this is a well-defined map, i.e., that if $\alpha \overset{\partial}{\simeq} \beta$ then $f \circ \alpha \overset{\partial}{\simeq} f \circ \beta$. This is clear: if $F\colon I \times I \to X$ is a homotopy with $f_0 = \alpha$ and $f_1 = \beta$, then $G = f \circ F$ is a homotopy from $f \circ \alpha$ to $f \circ \beta$. To show that f_* is a homomorphism, we need to show that $f_*([\alpha] \bullet [\beta]) = f_*([\alpha]) \bullet f_*([\beta])$. Since

$$f_*([\alpha] \bullet [\beta]) = f_*([\alpha * \beta]) = [f \circ (\alpha * \beta)]$$

and

$$f_*([\alpha]) \bullet f_*([\beta]) = [f \circ \alpha] \bullet [f \circ \beta] = [(f \circ \alpha) * (f \circ \beta)],$$

we are left with showing that $f \circ (\alpha * \beta) \overset{\partial}{\simeq} (f \circ \alpha) * (f \circ \beta)$. Note that, by definition of the concatenation of paths,

$$(f \circ \alpha) * (g \circ \beta) = \begin{cases} f \circ \alpha(2s) & s \leq 1/2 \\ f \circ \beta(2s-1) & s \geq 1/2 \end{cases}.$$

which is the same as the definition of $f \circ (\alpha * \beta)$. This completes the proof. $\square$

Example 12.6. Consider the covering map $p_2\colon (S^1, 1) \to (S^1, 1)$, $z \mapsto z^2$. Let $\omega_n\colon I \to S^1$ be the map $\omega_n(s) = \exp(2\pi i n s)$. Then $p_2 \circ \omega_n = \omega_{2n}$, and $(p_2)_*([\omega_n]) = ([\omega_{2n}])$. The induced map on $\mathbb{Z}$ is the doubling map

$$\begin{array}{ccc} \pi_1(S^1, 1) & \xrightarrow{(p_2)_*} & \pi_1(S^1, 1) \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{Z} & \xrightarrow{n \mapsto 2n} & \mathbb{Z} \end{array}$$

One similarly derives the induced map for $z \mapsto z^d$.

The next lemma shows that the fundamental group is a *functor*.

Lemma 12.7. *The induced homomorphism satisfies the properties:*

1. $(\mathrm{Id}_{(X,x_0)})_* = \mathrm{Id}_{\pi_1(X,x_0)}$*;*
2. *If* $f\colon (X, x_0) \to (Y, y_0)$ *and* $g\colon (Y, y_0) \to (Z, z_0)$*, then*

$$(g \circ f)_* = g_* \circ f_*$$

Proof. The first property is obvious: if nothing happens at the topological level, then nothing can happen at the algebraic level. For the second property, note that

$$(g \circ f)_*([\gamma]) = [g \circ f \circ \gamma] = g_*([f \circ g]) = (g_* \circ f_*)([\gamma]).$$

$\square$

An immediate consequence is that the fundamental group maps homeomorphic spaces to isomorphic groups. This allows to distinguish spaces: if two spaces X and Y have different fundamental group, they cannot be homeomorphic.

Corollary 12.8. *If* $f\colon (X, x_0) \to (Y, y_0)$ *is a homeomorphism, then* $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ *is a group isomorphism.*

Proof. We apply Lemma 12.7(1) and (2) with $g = f^{-1}$. Then

$$\mathrm{Id}_{\pi_1(X,x_0)} = (\mathrm{Id}_{(X,x_0)})_* = (f^{-1} \circ f)_* = (f^{-1})_* \circ f_*,$$

and similarly (reversing the role of f^{-1} and f) $\mathrm{Id}_{\pi_1(X,x_0)} = f_* \circ (f^{-1})_*$, which shows that $(f^{-1})_* = f_*^{-1}$. □

Example 12.9. Since $\pi_1(\mathbb{D}^2, 1) = \{0\}$ and $\pi_1(S^1, 1) \cong \mathbb{Z}$, $(\mathbb{D}^2, 1) \not\cong (S^1, 1)$.

In the following lectures we will see that this extends to *homotopy equivalence*.

12.2 Categories and functors

A **category** $\mathcal{C}$ consists of objects $\mathrm{obj}(\mathcal{C})$, for any ordered pair of objects (a, b) a set $\mathrm{Hom}_{\mathcal{C}}(a, b)$ whose elements are called morphisms or arrows (often written, $a \xrightarrow{f} b$) and composition maps $\mathrm{Hom}_{\mathcal{C}}(a, b) \times \mathrm{Hom}_{\mathcal{C}}(b, c) \to \mathrm{Hom}_{\mathcal{C}}(a, c)$, $(f, g) \mapsto g \circ f$, such that

1. (associativity) if $f \in \mathrm{Hom}_{\mathcal{C}}(a, b)$, $g \in \mathrm{Hom}_{\mathcal{C}}(b, c)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(c, d)$, then $h \circ (g \circ f) = (h \circ g) \circ f$;

2. (identity) for every $a \in \mathrm{obj}(\mathcal{C})$ there exists $\mathrm{id}_a \in \mathrm{Hom}_{\mathcal{C}}(a, a)$ such that $f \circ \mathrm{id}_a = \mathrm{id}_a \circ g$ for any $f \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(b, a)$.

In applications, the objects are often sets with a certain structures (vector spaces, topological spaces, groups) and the morphisms are structure-preserving maps between them (linear maps, continuous functions, group homomorphisms). While in these examples the objects are denoted by V, X, or G, the lower-case notation for objects in an arbitrary category indicates that there is no a priori requirement for these to be sets.

Let $\mathcal{C}$, $\mathcal{D}$ be two categories. A **functor** $F\colon \mathcal{C} \to \mathcal{D}$ assign to every object $a \in \mathrm{obj}(\mathcal{C})$ an object $F(a) \in \mathrm{obj}(\mathcal{D})$, and to every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ a morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(a), F(b))$, in such a way that

1. $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$;

2. if $f \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(b, c)$, then $F(g \circ f) = F(g) \circ F(f)$.

Example 12.10. Let Top_0 denote the category whose objects are pointed topological spaces (X, x_0), and whose morphisms are maps of pairs $(X, x_0) \xrightarrow{f} (Y, y_0)$. Let $\mathcal{G}$ be the category of groups, whose morphisms are group homomorphisms. Then the fundamental group π_1 is a functor:

$$\pi_1 \colon \mathrm{Top}_0 \to \mathcal{G}.$$

1. Every object (X, x_0) is assigned to a group $\pi_1(X, x_0)$;
2. Any map $(X, x_0) \xrightarrow{f} (Y, y_0)$ gives rise to a group homomorphism $\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0)$, where we write $f_* = \pi_1(f)$;
3. The identity gets mapped to the identity: $(\mathrm{Id}_{(X,x_0)})_* = \mathrm{Id}_{\pi_1(X,x_0)}$;
4. We have the property that $(g \circ f)_* = g_* \circ f_*$.

A functor as defined here is also called a *covariant* functor, because it preserves the direction of arrows. A *contravariant* functor is one that reverses the direction.

The language of categories and functors, sometimes also called "abstract nonsense", forms the basis of the modern treatment of many fields of mathematics, including algebraic geometry, number theory, and algebraic topology. It allows the study of structural similarities between mathematics concepts in an elegant way, and in particular it allows to transfer topological ideas to other fields of mathematics.