Lecture 13

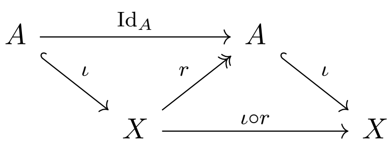
In the previous lecture we saw that the fundamental group is a functor: maps between pointed topological spaces get assigned to group homomorphisms in a way that preserves the identity map and compositions. We also saw that homeomorphisms correspond to isomorphisms in the category of groups. In this lecture we will study the effect of retractions, and deformation retracts, on the fundamental group.

13.1 Retractions

Let $A \subset X$ be topological spaces. Recall that a **retraction** is a map $r \colon X \to A$ such that $r|_A = \mathrm{Id}_A$.

Example 13.1. The map $\mathbb{C} - \{0\} \to S^1$, $z \mapsto |z|$, is a retraction.

A kind of converse to a retraction is the **inclusion** $\iota \colon A \to X$. We have the composition $r \circ \iota = \mathrm{Id}_A$, and the reverse composition $\iota \circ r \colon X \to X$. In diagrams,



where the arrow with hook $\hookrightarrow$ is used to emphasize that the map is *injective*, while the arrow with two tips $\twoheadrightarrow$ is used to emphasize that the map is *surjective*, or onto. A retract r is called a **deformation retract**, if

$$\iota \circ r \overset{A}{\simeq} \mathrm{Id}_X,$$

which means that there is a homotopy from $\iota \circ r$ to the identity Id_X that does nothing on A.

Proposition 13.2. *Let $A \subset X$, $r \colon X \to A$ a retractions, and $\iota \colon A \to X$ the inclusion. Let $x_0 \in A$ and let $r_* \colon \pi_1(X, x_0) \to \pi_1(A, x_0)$ and $\iota_* \colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ be the induced (push-forward) maps between the fundamental groups. Then:*

1. *r_* is surjective and ι_* is injective;*

2. *If r is a* deformation retract*, then r_* and ι_* are isomorphisms.*

Note that, in particular, the map

$$(\iota \circ r)_* \colon \pi_1(X, x_0) \to \pi_1(X, x_0)$$

is an isomorphism (though it may not be the identity, as is the case with $(r \circ \iota)_*$).

Proof. The first claim is clear: since $\mathrm{Id}_{\pi_1(A,x_0)} = (r \circ \iota)_* = r_* \circ \iota_*$, ι_* has to be injective (otherwise the composition couldn't be injective) and r_* has to be surjective (otherwise the composition couldn't be surjective).

To show that r_* is an isomorphism if r is a deformation retract, it is enough to show that r_* is injective. Let $[\gamma] \in \pi_1(X, x_0)$ and assume that $r_*([\gamma]) = [r \circ \gamma] = [e_A]$, or equivalently $r \circ \gamma \overset{\partial}{\simeq} e_A$, where e_A is the constant loop at x_0 *in* A. We need to show that in this case, $[\gamma] = [e_X]$, or $\gamma \overset{\partial}{\simeq} e_X$, where e_X is the constant loop at x_0 *in* X.

As $r \circ \gamma$ is a loop in $A \subset X$, $\iota \circ r \circ \gamma$ it is a loop in X, and $\iota \circ r \circ \gamma \overset{\partial}{\simeq} e_X$ by the same homotopy that takes $r \circ \gamma$ to e_A. By the transitivity of homotopy, it is therefore enough to show that

$$\iota \circ r \circ \gamma \overset{\partial}{\simeq} \gamma.$$

This follows by simply applying the homotopy from $\iota \circ r$ to Id_X to the loop γ. More precisely, let $F \colon X \times I \to X$ be the homotopy from $\iota \circ r$ to Id_X, so that $f_0 = \iota \circ r$ and $f_1 = \mathrm{Id}_X$. Construct a new homotopy $G \colon I \times I \to X$ by setting $G(s,t) = F(\gamma(s), t)$. Then $g_0 = \iota \circ r \circ \gamma$ and $g_1 = \gamma$. This concludes the proof. □

13.2 Applications

The first obvious application of Proposition 13.2 is that if $x_0 \in A \subset X$ and there is no surjective map from $\pi_1(X, x_0)$ to $\pi_1(A, x_0)$, then A cannot be a retract. Moreover, the the two fundamental groups are not isomorphic, then A cannot be a deformation retract of X.

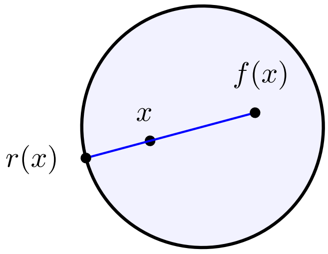
Recall that the closed unit disk was defined as

$$\mathbb{D}^2 = \{z \in \mathbb{C} \colon |z| \leq 1\}.$$

Proposition 13.3. *There is not retract $\mathbb{D}^2 \to S^1$.*

Proof. The existence of a retraction would imply a surjection $\pi_1(\mathbb{D}^2, 1) \twoheadrightarrow \pi_1(S^1, 1)$, but the fundamental group of the disk is $\pi_1(\mathbb{D}^1, 1) = \{0\}$, and the fundamental group of the circle is $\pi_1(S^1, 1) \cong \mathbb{Z}$. □

The following important result generalizes the fact that a continuous function $f \colon I \to I$ has a fixed point, i.e., a point $x \in I$ such that $f(x) = x$. In the case of the interval, this is an easy consequence of the intermediate value theorem. The generalization to maps from $I \times I$ to itself, or equivalently, from $\mathbb{D}^2$ to itself, is surprisingly non-trivial. A proof was found by Luitzen Egbertus Jan (Bertus) Brouwer in 1910.



Theorem 13.4. *(Brouwer Fixed Point Theorem) Every map $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point.*

Proof. The proof is by contradiction. Assume that the statement is wrong, and that there is a map $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ such that $f(x) \neq x$ for all $x \in \mathbb{D}^2$. For every $x \in \mathbb{D}^2$, there is a unique line joining through x and $f(x)$, parametrized by $L_x(t) = (1-t)f(x) + tx$ for $t \in \mathbb{R}$. This line intersects the boundary circle S^1 in exactly in two points, one for which $t > 0$. Denote this point by $r(x)$.

We thus get a map $r\colon \mathbb{D}^2 \to S^1$ such that $r(x) = x$ for $x \in S^1$. We next show that this map is continuous, and thus gives a retraction. Indeed, the function r is given by $r(x) = L_x(t_*)$, where t_* is the positive solution to the quadratic equation

$$|(1-t)f(x) + tx|^2 = 1.$$

Writing this out, we get a quadratic equation with precisely two solutions, only one of which is positive. From the explicit expression for the solution of a quadratic equation, it follows that such a t_* depends continuously on the coefficients of the equation, which in turn depend continuously on x and $f(x)$. It follows that $r\colon \mathbb{D}^2 \to S^1$ is continuous. □

Exercise 13.5. Show that Theorem 13.4 still holds if we replace $(\mathbb{D}^2, S^1)$ with a pair of spaces (X, A) such that $A \subset X$, $X \cong \mathbb{D}^2$, and $A \cong S^1$. Hence, Theorem 13.4 also holds for maps $f \times I^2 \to I^2$ (where $I^2 = I \times I$ is the square).