We will have a closer look at the construction of topological spaces using disjoint unions and quotient spaces, and show how to formalize “cut and paste” operations.

### 2.1 New spaces from old

**Definition 2.1.** Let $X, Y$ be topological spaces. The **disjoint union** of $X$ and $Y$ is the topological space with underlying set

$$X \sqcup Y = X \times \{0\} \cup Y \times \{1\},$$

and topology whose open sets are generated by sets of the form $U \times \{0\}$ and $V \times \{1\}$ for $U \subset X$ and $V \subset Y$ open.

**Remark 2.2.** The $\{0\}$ and $\{1\}$ in the definition are rather arbitrary. Taking the product of a topological space with a point does not change the topology, but it ensures that the two spaces do not overlap. For example, $X \sqcup X$ amounts to taking the union of $X$ and a disjoint copy of $X$, while $X \cup X$ is just $X$.

**Example 2.3.** The 0-sphere $S^0 \cong \{\text{pt}\} \sqcup \{\text{pt}\}$ (recall that pt is short for any point).

Another important construction is the quotient, which formalizes the notion of “gluing”. Recall that an equivalence relation is a subset $E \subset X \times X$ such that:

- for all $x \in X$, $(x, x) \in E$ (reflexive);
- if $(x, y) \in E$, then $(y, x) \in E$ (symmetric);
- if $(x, y) \in E$ and $(y, z) \in E$, then $(x, z) \in E$ (transitive).

Once we fix an equivalence relation $E$, we usually write $x \sim y$ instead of $(x, y) \in E$. The **equivalence class** of $x \in X$ is the set

$$[x] = \{y \in X \mid x \sim y\}.$$

The set of all equivalence classes of an equivalence relation $E$ is denoted by $X/E$ or $X/\sim$. The **quotient map** is the map $q: X \to X/E$, $q(x) = [x]$. 
**Definition 2.4.** The quotient topology on $X/E$ has as open sets those $V \subset X/E$ for which $q^{-1}(V) = \{x \in X \mid q(x) \in V\}$ is open.

Note that a subset $V \subset X/E$ is open if and only if

$$U = \bigcup_{[x] \in V} [x]$$

is open in $X$. By definition, the quotient map is continuous.

**Exercise 2.5.** Let $X$ be a topological space, $E$ an equivalence relation, and $X/E$ the corresponding quotient space. Show that for all topological spaces $Z$ and all functions $g: X/E \to Z$, $g$ is continuous if and only if the composition $f = g \circ q$ is continuous.

![Diagram](image)

**Example 2.6.** Consider $X = [0, 1] =: I^1$. Define the equivalence relation

$$x \sim y \Leftrightarrow (x = y) \text{ or } (x = 0, y = 1) \text{ or } (x = 1, y = 0).$$

The quotient space $X/E$ then consists of the classes $[x] = \{x\}$ for $x \notin \{0, 1\}$ and $[0] = [1] = \{0, 1\}$.

![Diagram](image)

The result is an interval with the endpoints “glued together”, sometimes written $[0, 1)/(0 \sim 1)$ to highlight the fact that only 0 and 1 are identified. If we parametrize the circle by

$$f: [0, 1] \to S^1, \quad t \mapsto (\cos(2\pi t), \sin(2\pi t)),$$

then this map is one-to-one except at the endpoints, and identifying these endpoints gives rise to a homeomorphism. One also says that the map $f$ “factoring” over the
quotient space, as indicated in the following diagram:

\[ [0, 1] \quad \xrightarrow{q} \quad [0, 1]/(0 \sim 1) \cong S^1 \]

The above example is a special case of a more general construction. Let \( A \subset X \) be a subset. Such a subset gives rise to the equivalence relation

\[ x \sim y \Leftrightarrow (x = y) \text{ or } \{x, y\} \subset A. \]

The corresponding quotient space, by some abuse of notation sometimes referred to as \( X/A \) (note that \( A \) is a subset of \( X \), not \( X \times X \)), consists of classes \( [x] = \{x\} \) if \( x \in X - A \) and \( [x] = [y] \) if \( x, y \in A \). In words, \( X/A \) corresponds to “crushing” the set \( A \) onto one point.

Exercise 2.7. Recall the set inclusion \( S^{n-1} \subset \mathbb{B}^n \). Show that \( \mathbb{B}^n/S^{n-1} \cong S^n \). Since \( [0, 1] \cong [-1, 1] = \mathbb{B}^1 \) and \( \{0, 1\} \cong \{-1, 1\} = S^0 \), this generalizes Example 2.6. Can you interpret the case \( n = 2 \) visually?

Disjoint unions and quotients can be combined to construct new spaces.

Example 2.8. Consider the disjoint union of \( \mathbb{Z} \) times the interval \([0, 1] \). Formally, this amounts to taking the union

\[ X = \bigcup_{k \in \mathbb{Z}} [0, 1] \times \{k\}. \]

By gluing the endpoints, i.e., identifying \( \{1\} \times \{k\} \) with \( \{0\} \times \{k + 1\} \), one obtains a set that is homeomorphic to \( \mathbb{R} \) (check this!)

In a similar fashion, one can build up \( \mathbb{R}^2 \) by tiling more sophisticated shapes.
Example 2.9. Consider the square $X = I^2 = [0, 1]^2$. Define an equivalence relation

$$(x, y) \sim (x', y') \iff (x, y) = (x', y') \text{ or } (y = y', \{x, x'\} \subset \{0, 1\}).$$

In words, we identify each point on the left boundary with the corresponding point on the right boundary. The result is a space homeomorphic to a cylinder.

Example 2.10. Consider again the square and define an equivalence relation by

$$(x, y) \sim (x', y') \iff (x, y) = (x', y') \text{ or } (y = 1 - y', \{x, x'\} \subset \{0, 1\}).$$

We now identify each point on the left boundary with coordinate $y$ with the point on the right boundary with coordinate $1 - y$. The result of this construction is the famous Möbius strip, a surface with only one side. This surface is not homeomorphic to the cylinder (try to find out why!)

Exercise 2.11. Show that

$$\mathbb{R}^2/\mathbb{Z}^2 \cong T^2,$$

where $\mathbb{R}^2/\mathbb{Z}^2$ is the quotient space with respect to the equivalence relation

$$(x, y) \sim (x', y') \iff x - x' \in \mathbb{Z}, \ y - y' \in \mathbb{Z}.$$