Lecture 2

We will have a closer look at the construction of topological spaces using disjoint unions and quotient spaces, and show how to formalize "cut and paste" operations.

2.1 New spaces from old

Definition 2.1. Let X, Y be topological spaces. The **disjoint union** of X and Y is the topological space with underlying set

$$X \sqcup Y = X \times \{0\} \cup Y \times \{1\},$$

and topology whose open sets are generated by sets of the form $U \times \{0\}$ and $V \times \{1\}$ for $U \subset X$ and $V \subset Y$ open.

Remark 2.2. The $\{0\}$ and $\{1\}$ in the definition are rather arbitrary. Taking the product of a topological space with a point does not change the topology, but it ensures that the two spaces do not overlap. For example, $X \sqcup X$ amounts to taking the union of X and a disjoint copy of X, while $X \cup X$ is just X.

Example 2.3. The 0-sphere $S^0 \cong \{\mathrm{pt}\} \sqcup \{\mathrm{pt}\}$ (recall that pt is short for any point).

Another important construction is the quotient, which formalizes the notion of "gluing". Recall that an equivalence relation is a subset $E \subset X \times X$ such that:

- for all $x \in X$, $(x, x) \in E$ (reflexive);
- if $(x, y) \in E$, then $(y, x) \in E$ (symmetric);
- if $(x, y) \in E$ and $(y, z) \in E$, then $(x, z) \in E$ (transitive).

Once we fix an equivalence relation E, we usually write $x \sim y$ instead of $(x, y) \in E$. The **equivalence class** of $x \in X$ is the set

$$[x] = \{y \in X \mid x \sim y\}.$$

The set of all equivalence classes of an equivalence relation E is denoted by ${}^{X}\!/_{E}$ or ${}^{X}\!/_{\sim}$. The **quotient map** is the map $q\colon X \to {}^{X}\!/_{E}$, $q(x) = [x]$.

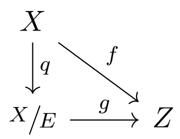
Definition 2.4. The **quotient topology** on ${}^{X}/_{E}$ has as open sets those $V \subset {}^{X}/_{E}$ for which $q^{-1}(V) = \{x \in X \mid q(x) \in V\}$ is open.

Note that a subset $V \subset {}^{X}/_{E}$ is open if and only if

$$U = \bigcup_{[x] \in V} [x]$$

is open in X. By definition, the quotient map is continuous.

Exercise 2.5. Let X be a topological space, E an equivalence relation, and ${}^{X}/_{E}$ the corresponding quotient space. Show that for all topological spaces Z and all functions $g\colon {}^{X}/_{E} \to Z$, g is continuous if and only if the composition $f = g \circ q$ is continuous.



Example 2.6. Consider $X = [0,1] =: I^1$. Define the equivalence relation

$$x \sim y \Leftrightarrow (x = y) \text{ or } (x = 0, y = 1) \text{ or } (x = 1, y = 0).$$

The quotient space ${}^{X}/_{E}$ then consists of the classes $[x] = \{x\}$ for $x \notin \{0,1\}$ and $[0] = [1] = \{0,1\}$.

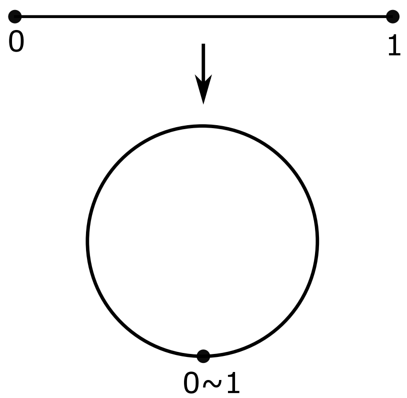


Figure 2.1: Glueing an interval at the endpoints to obtain a circle.

The result is an interval with the endpoints "glued together", sometimes written ${}^{[0,1]}/_{(0 \sim 1)}$ to highlight the fact that only 0 and 1 are identified. If we parametrize the circle by

$$f\colon [0,1] \to S^1, \quad t \mapsto (\cos(2\pi t), \sin(2\pi t)),$$

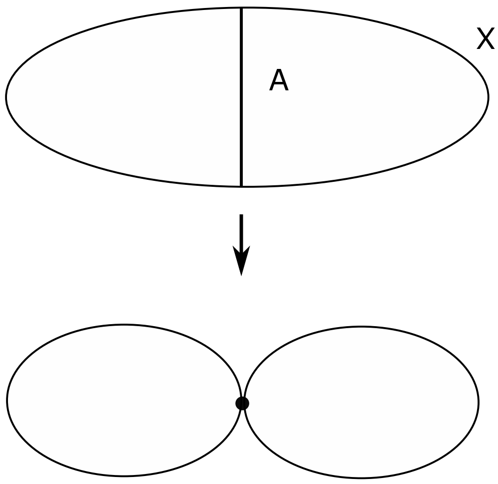
then this map is one-to-one except at the endpoints, and identifying these endpoints gives rise to a homeomorphism. One also says that the map f "factoring" over the quotient space, as indicated in the following diagram:

$$\begin{array}{ccc} [0,1] & & \\ \downarrow q & \searrow f & \\ {}^{[0,1]}/_{(0\sim 1)} & \xrightarrow{\cong} & S^1 \end{array}$$

The above example is a special case of a more general construction. Let $A \subset X$ be a subset. Such a subset gives rise to the equivalence relation

$$x \sim y \Leftrightarrow (x = y) \text{ or } \{x, y\} \subset A.$$

The corresponding quotient space, by some abuse of notation sometimes referred to as ${}^{X}/_{A}$ (note that A is a subset of X, not $X \times X$!), consists of classes $[x] = \{x\}$ if $x \in X - A$ and $[x] = [y]$ if $x, y \in A$. In words, ${}^{X}/_{A}$ corresponds to "crushing" the set A onto one point.



Exercise 2.7. Recall the set inclusion $S^{n-1} \subset \mathbb{B}^n$. Show that ${}^{\mathbb{B}^n}/_{S^{n-1}} \cong S^n$. Since $[0,1] \cong [-1,1] = \mathbb{B}^1$ and $\{0,1\} \cong \{-1,1\} = S^0$, this generalizes Example 2.6. Can you interpret the case $n = 2$ visually?

Disjoint unions and quotients can be combined to construct new spaces.

Example 2.8. Consider the disjoint union of $\mathbb{Z}$ times the interval $[0,1]$. Formally, this amounts to taking the union

$$X = \bigcup_{k\in\mathbb{Z}} [0,1] \times \{k\}.$$

By gluing the endpoints, i.e., identifying $\{1\} \times \{k\}$ with $\{0\} \times \{k+1\}$, one obtains a set that is homeomorphic to $\mathbb{R}$ (check this!)

In a similar fashion, one can build up $\mathbb{R}^2$ by tiling more sophisticated shapes.

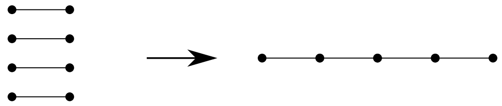


Figure 2.2: The real line by gluing intervals.

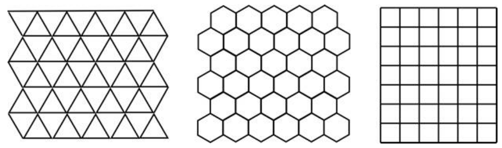
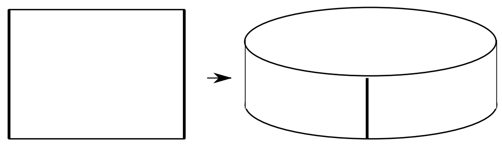


Figure 2.3: The plane by tiling shapes.

Example 2.9. Consider the square $X = I^2 = [0,1]^2$. Define an equivalence relation

$$(x,y) \sim (x',y') \Leftrightarrow (x,y) = (x',y') \text{ or } (y = y', \{x,x'\} \subset \{0,1\}).$$

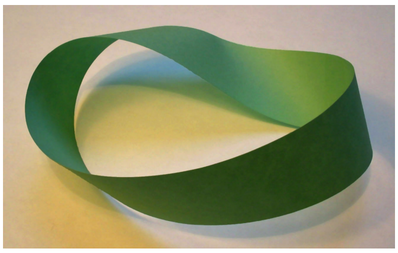
In words, we identify each point on the left boundary with the corresponding point on the right boundary. The result is a space homeomorphic to a cylinder.



Example 2.10. Consider again the square and define an equivalence relation by

$$(x,y) \sim (x',y') \Leftrightarrow (x,y) = (x',y') \text{ or } (y = 1-y', \{x,x'\} \subset \{0,1\}).$$

We now identify each point on the left boundary with coordinate y with the point on the right boundary with coordiante $1-y$. The result of this construction is the famous



Möbius strip, a surface with only one side. This surface is *not* homeomorphic to the cylinder (try to find out why!)

Exercise 2.11. Show that

$$\mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{T}^2,$$

where $\mathbb{R}^2/\mathbb{Z}^2$ is the quotient space with respect to the equivalence relation

$$(x,y) \sim (x',y') \Leftrightarrow x - x' \in \mathbb{Z},\ y - y' \in \mathbb{Z}.$$