Lecture 3

So far we looked at the notion of homeomorphism, and considered spaces to be "topologically equivalent" if they are homeomorphic. Intuitively, homeomorphic spaces are the same up to "stretching" and "shrinking", but not crushing or cutting. Homeomorphism is a rather fine equivalence, and considering a coarser relation such as homotopy equivalence can be useful. We begin by discussing retractions, continuous functions of a space to a subspace, and deformation retracts, which formalize the idea of *continuously* squeezing a space onto a subspace. This concept will lead naturally to the idea of homotopy.

3.1 Retractions

Definition 3.1. A **pair** of spaces (X, A) consists of a topological space X and a subspace $A \subset X$. If $A = \{x\}$, then we write (X, x) and call this a **pointed** space.

Example 3.2. Consider the pair of spaces $(\mathbb{R}^2 - \{0\}, S^1)$, or the pair $(S^1, (1, 0))$.

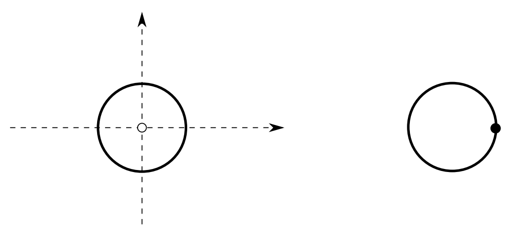

Figure 3.1: The pairs $(\mathbb{R}^2 - \{0\}, S^1)$ and $(S^1, (1, 0))$.

Definition 3.3. A subset $A \subset X$ is a **retract** of X if there is a map $r\colon X \to A$ (the **retraction**) such that the restriction satisfies

$$r|_A = \mathrm{Id}_A,$$

i.e., $r(a) = a$ for $a \in A$.

Example 3.4. The set $\mathbb{R}^2 - \{0\}$ retracts to S^1 via $r(x) = x/\|x\|$.

Exercise 3.5. Show that $X = [0, 1]$ *does not* retract to $A = \{0, 1\}$.

The following generalization is non-trivial.

Theorem 3.6. *(Brouwer) The ball $\mathbb{B}^n$ does not retract to S^{n-1}.*

We next describe what it means to *deform* a space onto a subspaces in a continuous manner.

Definition 3.7. Let (X, A) be a pair of spaces. X **deformation retracts** to A (and A is called a **deformation retract** of X), if there exists a one-parameter family of functions $f_t\colon X \to X$, $t \in I = [0,1]$, such that

$$f_0 = \mathrm{Id}_X, \quad f_1(X) = A, \quad f_t|_A = \mathrm{Id}_A, t \in [0,1],$$

and the map $X \times I \to X$, $(x,t) \mapsto f_t(x)$ is continuous.

In the literature, this notion is sometimes called a *strong* deformation retract, the strong referring to the requirement that $f_t|_A = \mathrm{Id}_A$ throughout.

Example 3.8. $\mathbb{R}^n$ deformation retracts to 0 by means of $f_t(x) = (1-t)x$.

Exercise 3.9. Show that $\mathbb{R}^n - \{0\}$ deformation retracts to S^{n-1} via $f_t(x) = (1-t)x + tx/\|x\|$.

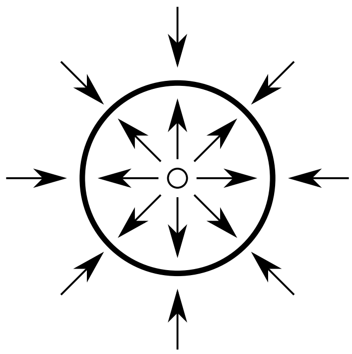

Figure 3.2: Deformation of a punctured plane $\mathbb{R}^2 - \{0\}$ onto the circle S^1.

3.2 Homotopy

Definition 3.10. Let X, Y be topological spaces, and $I = [0,1]$. A map

$$F\colon X \times I \to Y$$

is called a **homotopy**. If $f_t(x) = F(x,t)$, then F is called a homotopy from f_0 to f_1. We say that two maps f, g are **homotopic**, written $f \simeq g$, if there exists a homotopy F such that $f_0 = f$ and $f_1 = g$.

Proposition 3.11. *Homotopy is an equivalence relation: if $f, g, h\colon X \to Y$ are maps, then* *(i)* $f \simeq f$*;*

(ii) $f \simeq g \Leftrightarrow g \simeq f$*;*

(iii) $f \simeq g, g \simeq h \Rightarrow f \simeq h$*.*

The proof requires the following observation.

Lemma 3.12. *(Pasting Lemma) Let* $X = A \cup B$*, with* A, B *both closed subspaces of a topological space* X*. Let* $f\colon X \to Y$ *be a function and assume that* $f|_A$ *and* $f|_B$ *are continuous. Then* f *is continuous.*

Exercise 3.13. Prove Lemma 3.12.

Proof of Proposition 3.11. Exercise (2.2). □

Definition 3.14. Let X, Y be topological spaces. We say X is **homotopy equivalent** to Y (written $X \simeq Y$), if there are maps

$$f\colon X \to Y, \quad g\colon Y \to X,$$

such that

$$g \circ f \simeq \mathrm{Id}_X, \quad f \circ g \simeq \mathrm{Id}_Y.$$

Homotopy equivalence allows for crushing spaces, but not for tearing. Note that homeomorphic spaces are homotopy equivalent, but the converse does not hold. In particular, if we manage two show that two spaces are not homotopy equivalent, then they cannot be homeomorphic.

Definition 3.15. A topological space X is called **contractible** if it is homotopy equivalent to a point.

Example 3.16. Any $\mathbb{R}^n$ is homotopy equivalent to a point, $\mathbb{R}^n \simeq \mathbb{R}^0$. To see this, consider the maps

$$f\colon \mathbb{R}^n \to \mathbb{R}^0, \quad x \mapsto 0$$

and

$$g\colon \mathbb{R}^0 \to \mathbb{R}^n, \quad 0 \mapsto 0.$$

Note that $f \circ g = \mathrm{Id}_{\mathbb{R}^0}$. Consider now the map $g \circ f\colon \mathbb{R}^n \to \mathbb{R}^n$, mapping any x to 0. Consider the straight-line homotopy

$$F\colon \mathbb{R}^n \times I \to \mathbb{R}^n, \quad (x,t) \mapsto tx.$$

Then $f_0 = g \circ f$ and $f_1 = \mathrm{Id}_{\mathbb{R}^n}$, which shows that $g \circ f \simeq \mathrm{Id}_{\mathbb{R}^n}$.

Exercise 3.17. Show that $\simeq$ is an equivalence relation on topological spaces.

Example 3.18. For all m, n, we have $\mathbb{R}^n \simeq \mathbb{R}^m$. More generally, for all topological spaces X, we have $X \times \mathbb{R}^n \simeq X$.

Example 3.19. We have $\mathbb{R}^m - \{0\} \simeq S^{m-1}$. To construct the homotopy, consider the inclusion map $f\colon S^{m-1} \to \mathbb{R}^m - \{0\}$, and the retraction $g\colon \mathbb{R}^m - \{0\} \to S^{m-1}$, given by $x \mapsto x/\|x\|$. Then $g \circ f = \mathrm{Id}_{S^{m-1}}$, and for $f \circ g$ we construct the homotopy

$$F\colon \mathbb{R}^m - \{0\} \times I \to \mathbb{R}^m - \{0\}, \quad (x,t) \mapsto (1-t)x + tx/\|x\|.$$

Note that this is just the deformation retract from Example 3.9, and that $f_0 = \mathrm{Id}_{\mathbb{R}^n - \{0\}}$ and $f_1 = g \circ f$.

How does homotopy relate to the notion of a deformation retract? Clearly, if a space X deformation retracts to a point $x \in X$, then it is **contractible** (homotopy equivalent to a point), but the converse need not hold!

Example 3.20. It is non-trivial to show that spheres S^n for $n \geq 1$ are not contractible.