Lecture 4

We have seen different notions of equivalence: homeomorphism ($X \cong Y$), homotopy of maps ($f \simeq g$) and homotopy equivalence of spaces ($X \simeq Y$). The last of these notions allows spaces to be identified that look superficially different, but can be somehow deformed or continuously collapsed into one another. In this lecture we first review the notion of contractible spaces (already mentioned in Lecture 3) and then study paths and loops. Recall that a **map** is always assumed to be continuous.

4.1 Contractible spaces

Definition 4.1. A topological space X is called **contractible** if $X \simeq \{\mathrm{pt}\}$.

We have seen that $\mathbb{R}^n$ for $n \geq 1$ is contractible: there exist maps $f \colon \mathbb{R}^n \to \mathbb{R}^0$ and $g \colon \mathbb{R}^0 \to \mathbb{R}^n$ such that $f \circ g = \mathrm{Id}_{\mathbb{R}^0}$ and $g \circ f \simeq \mathrm{Id}_{\mathbb{R}^n}$, with the homotopy from the identity $\mathrm{Id}_{\mathbb{R}^n}$ to the retraction $g \circ f \colon \mathbb{R}^n \to \{0\}$ given by the *linear homotopy* $F(x,t) = (1-t)x$.

Remark 4.2. One should compare the notion of contractibility with that of a deformation retract. If X deformation retracts to a point $A = \{a\} \subset X$, then X is contractible. To be more precise, by the definition of a deformation retract, we have a family of maps $f_t \colon X \to X$ such that $f_0 = \mathrm{Id}_X$, $f_1(x) = a$ for $x \in X$, and $f_t(a) = a$ for all t. Considering the retraction $f \colon X \to A$ and the inclusion $g \colon A \to X$, we get $f \circ g = \mathrm{Id}_A$. On the other hand, $g \circ f = f_1$ and $\mathrm{Id}_X = f_0$, so that we get a homotopy $\mathrm{Id}_X \simeq g \circ f$ by setting $F(x,t) = f_t(x)$.

The converse is **not true**: a spaces can be contractible but not deformation retract to a point (try to think of an example!)

Theorem 4.3. *The sphere S^n is not contractible.*

Exercise 4.4. Proof the above Theorem for $n = 0$. The case $n = 1$ will be shown later.

4.2 Paths

Definition 4.5. Let $x, y \in X$. A **path** from x to y is a map $f \colon I \to X$ with $f(0) = x$ and $f(1) = y$.

Definition 4.6. Let $f, g\colon I \to X$ be paths with $f(1) = g(0)$. The path $f * g\colon I \to X$, defined by

$$f * g(t) = \begin{cases} f(2t) & t \leq 1/2 \\ g(2t-1) & t \geq 1/2. \end{cases}$$

is called the **concatenation** of f and g.

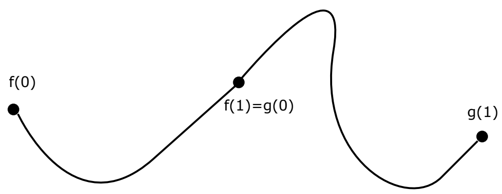

Figure 4.1: The concatenation of two paths.

Definition 4.7. A topological space X is called **path connected** if for any two points $x, y \in X$ there exists a path $f\colon I \to X$ with $f(0) = x$ and $f(1) = y$.

Since paths are maps between topological spaces, we can consider *homotopies* of paths: given two paths $f, g\colon I \to X$, a homotopy is given by a map $F\colon I \times I \to X$ with $f_0 = f$ and $f_1 = g$. It is not hard to show that if X is path connected, then every path $f\colon I \to X$ is homotopic to a constant path $g(t) = x$ (consider a path $g_t\colon I \to X$ from any point $f(t)$ to x, and define the homotopy $F(s,t) = g_t(s)$). To get more useful topological information out of paths, we consider paths with common endpoints.

Definition 4.8. Let $x, y \in X$ and let $f, g\colon I \to X$ be paths from x to y. Then f is homotopic to g relative to the boundary (or relative to the endpoints), written $f \overset{\partial}{\simeq} g$, if there is a homotopy

$$F\colon I \times I \to X$$

with $f_0 = f$, $f_1 = g$ and for all t, $f_t(0) = x$, $f_t(1) = y$.

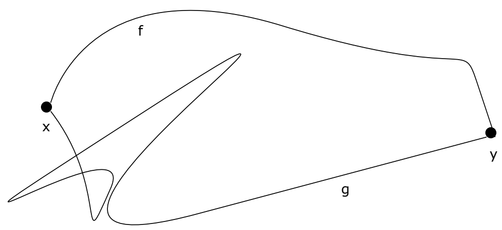

Figure 4.2: Homotopic relative to endpoints.

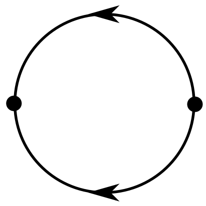

Figure 4.3: Two paths that are homotopic, but where there is no homotopy in S^1 that preserves endpoints.

Example 4.9. Let $X = S^1 \subset \mathbb{C}$ and consider the maps $f(t) = \exp(\pi i t)$ and $g(t) = \exp(-\pi i t)$. Thus $f(t)$ moves from 1 to -1 along the top, while $g(t)$ moves from 1 to -1 along the bottom half of the circle.

Then $f \simeq g$, but not in an end-point preserving fashion. While constructing a homotopy between these paths is easy, showing that this can't be done in an end-point preserving way is surprisingly hard!

Lemma 4.10. *Let $x, y \in X$. The relative homotopy $\overset{\partial}{\simeq}$ is an equivalence relation on the set of paths $I \to X$ with endpoints x, y.*

Proof. It is clear that $f \overset{\partial}{\simeq} f$ and $f \overset{\partial}{\simeq} g \Leftrightarrow g \overset{\partial}{\simeq} f$ for paths $f, g \colon I \to X$ with common endpoints. To show transitivity, let $f, g, h \colon I \to X$ be paths from x to y such that $f \overset{\partial}{\simeq} g$ and $g \overset{\partial}{\simeq} h$. This means that there are homotopies

$$F \colon I \times I \to X, \quad G \colon I \times I \to X$$

such that $f_0 = f$, $f_1 = g_0 = g$, and $g_1 = h$. Define a new map $H \colon I \times I \to X$ by

$$H(s,t) = \begin{cases} F(s, 2t) & \text{if } t \leq 1/2 \\ G(s, 2t-1) & \text{if } t \geq 1/2 \end{cases}.$$

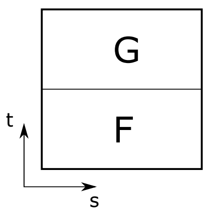

Figure 4.4: The homotopy H coincides with (a reparametrized version of) F on the lower rectangle ($t \leq 1/2$), and with G on the upper rectangle ($t \geq 1/2$).

Clearly, $h_0 = f$ and $h_1 = h$. Moreover, by the Pasting Lemma, H is continuous, which shows that $f \overset{\partial}{\simeq} h$. ∎

Lemma 4.11. *Assume that $f \overset{\partial}{\simeq} g$ and $f' \overset{\partial}{\simeq} g'$, where $f, g\colon I \to X$ are paths with $f(1) = g(0)$. Then $f * f' \overset{\partial}{\simeq} g * g'$.*

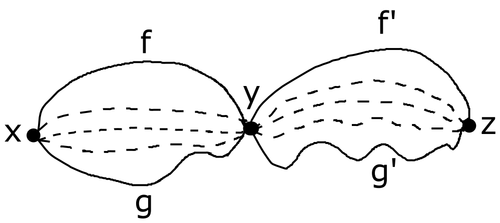

Figure 4.5: Concatenated homotopies.

Proof. The proof is essentially the same as that of Lemma 4.10, but with the role of s and t reversed. The situation is visualized as follows.

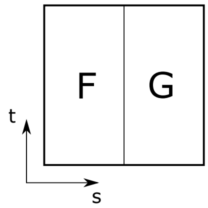

Figure 4.6: The homotopy H coincides with (a reparametrized version of) F on the left rectangle ($s \leq 1/2$), and with G on the right rectangle ($s \geq 1/2$).

Formally, consider homotopies F and F' with $f_0 = f$, $f_1 = g$, $f'_0 = f'$, $f'_1 = g'$. Define a new map

$$G(s,t) = \begin{cases} F(2s,t) & \text{if } s \leq 1/2 \\ G(2s-1,t) & \text{if } s \geq 1/2 \end{cases}.$$

As before, this map is continuous, and satisfies $h_0 = f * f'$ and $h_1 = g * g'$, thus showing that $f * f' \overset{\partial}{\simeq} g * g'$. □

In the coming lecture we will look at special types of paths, called **loops**, which start and end at the same point. Using Lemma 4.10 and 4.11, we will see that the set of equivalence classes of loops have a group structure, leading to the concept of the **Fundamental Group** of a pointed topological space.