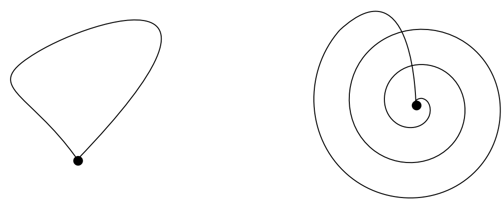
# Lecture 5

In this lecture we look at loops and will discover that there is an underlying algebraic structure, the Fundamental Group, that allows to gain insight into the topological features of spaces.

## 5.1 Loops and the Fundamental Group

We call a topological space $X$ with a point $x_0$ a **pointed space** $(X, x_0)$.

**Definition 5.1.** Let $(X, x_0)$ be a pointed space. A **loop** is a path $f\colon I \to X$ with $f(0) = f(1) = x_0$.



Note that the composition of two loops is again a loop.

Since in Lecture 4 we have seen that homotopy on paths with common endpoints is an equivalence relation, we can form equivalence classes of loops

$$[f] = \{g\colon g\colon I \to X, g(0) = g(1) = x_0, g \overset{\partial}{\simeq} f\}.$$

The set of equivalence classes of loops on $(X, x_0)$ is denoted by $\pi_1(X, x_0)$. We have seen that if $f \simeq g$ and $f' \simeq g'$, then $f * f' \simeq g * g'$. From this it follows that the equivalence class $[f * f']$ only depends on the classes $[f]$ and $[f']$, and not on the particular choice of representative in each class. This allows us to define a product

$$[f] \bullet [g] := [f * g].$$

**Proposition 5.2.** $(\pi_1(X, x_0), \bullet)$ *is a group, called the* ***Fundamental Group*** *of the pointed space* $(X, x_0)$*. The unit element is the class* $[e]$ *of the constant loop, and for every* $[f]$*, the inverse* $[f]^{-1}$ *is the class* $[\overline{f}]$*, where* $\overline{f}(t) = f(1-t)$ *is the inverse loop.*

*Proof.* **(1)** We first show that $[f] \bullet [e] = [e] \bullet [f] = [f]$. If $f\colon I \to X$ is a loop, we thus need to show that

$$f * e \overset{\partial}{\simeq} e * f \overset{\partial}{\simeq} f.$$

For this, we first construct a homotopy from $f * e$ to $e * f$ as follows

$$F(s,t) = \begin{cases} x_0 & 2s \leq t \text{ or } 2s-1 \geq t \\ f(2s-t) & \text{else.} \end{cases}$$

Indeed, we see that

$$f_0(s) = F(s,0) = \begin{cases} f(2s) & \text{if } s \leq 1/2 \\ x_0 & \text{if } s \geq 1/2, \end{cases}$$

which is the definition of $f * e$. Similarly, one checks that $f_1$ coincides with $e * f$. By the Pasting Lemma, we get a homotopy. How does one derive this homotopy? A simple way is to draw a diagram to visualise what is happening.

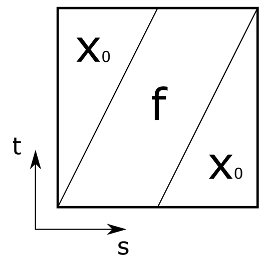



Figure 5.1: The composition of $f$ and the constant loop on the lower boundary, and the composition of the constant loop and $f$ on the upper boundary. The parameter $t$ on the vertical axis parametrizes the different maps $f_t$ in the homotopy.

In Figure 5.1, one then only needs to figure out the regions in $(s,t)$ space where the homotopy coincides with $f$, and where it coincides with $x_0$. Of course, the functions should then be rescaled so that every vertical slice of of the $f$-band starts with $f(0)$ and ends with $f(1)$. To show that $f \overset{\partial}{\simeq} e * f$, one similarly uses a diagram

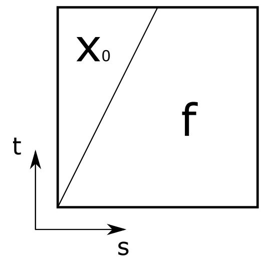



Figure 5.2: The loop $f$ is homotopic to the composition $e * f$.

We leave the explicit description (and that for $f \overset{\partial}{\simeq} f * e$) as an exercise.

**(2)** We next show the existence of the inverse. Let $f\colon I \to X$ be a loop and $\overline{f}\colon I \to X$ the loop with $\overline{f}(s) = f(1-s)$. We need to show that $\overline{f} * f \overset{\partial}{\simeq} e$ and $f * \overline{f} \overset{\partial}{\simeq} e$. For this, we consider the diagram

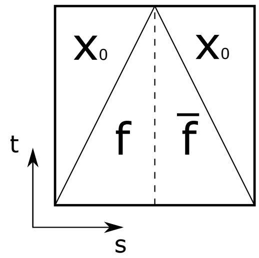



Figure 5.3: A loop composed with its inverse is homotopic to the constant loop.

Explicitly, this diagram suggests the homotopy

$$F(s,t) = \begin{cases} x_0 & \text{if } 2s \leq t \text{ of } 2-2s \leq t \\ f(2s-t) & \text{if } 2s \leq 1 \text{ and } t \leq 2s \\ f(2s+t-1) & \text{if } 2s \geq 1 \text{ and } t \leq 2s-2. \end{cases}$$

One verifies that $f_0 = f * \overline{f}$ and that $f_1 = e$. The same diagram with $\overline{f}$ and $f$ interchanged gives a homotopy $\overline{f} * f \overset{\partial}{\simeq} e$.

**(3)** To verify associativity, namely that $(f * g) * h \overset{\partial}{\simeq} f * (g * h)$, we use the diagram

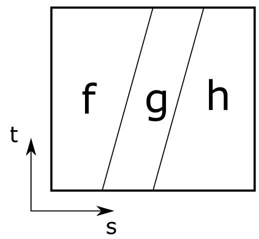



Figure 5.4: Three loops composed in different order.

The detailed expression for the homotopy is left as an exercise. □

**Example 5.3.** $\pi_1(\mathbb{R}^2, \{0\}) = 0$, as every loop is homotopic to the origin.

**Example 5.4.** Let $X = S^1 \subset \mathbb{C}$ with $x_0 = 1$. Then $\pi_1(X, x_0) \cong \mathbb{Z}$: every loop is of the form $f(t) = \exp(2\pi kit)$ for $k \in \mathbb{Z}$, that is, it goes around $k$ times in clockwise or anti-clockwise direction. The composition of two loops is clearly another loop. On the unit circle, this composition can be realized as the product $\exp(2\pi kit) \cdot \exp(2\pi mit) = \exp(2\pi i(k+m)t)$. While this example is intuitively clear, it is not a formal proof! We will get back to this example in Lectures 6 to 8, when we discuss covering spaces.