Lecture 6

6.1 Path connected spaces

For certain spaces, $\pi_1(X, x_0)$ is actually a topological invariant of the space itself: it does not depend on the choice of basepoint.

Proposition 6.1. *If X is path-connected, then for any two $x_0, x_1 \in X$, the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Proof. Let $h\colon I \to X$ be a path from x_0 to x_1, with inverse path $\overline{h}$. Define the map

$$\begin{aligned} \beta_h\colon \pi_1(X, x_0) &\to \pi_1(X, x_1) \\ [f] &\mapsto [\overline{h} * f * h]. \end{aligned}$$

We need to show that β_h is an isomorphism of groups, with $\beta_{\overline{h}} = \beta_h^{-1}$ as inverse.

(1) We first show that β_h is a bijection. Note that since $h * \overline{h} \overset{\partial}{\simeq} e_{x_0}$ (the constant loop on x_0) and $\overline{h} * h \overset{\partial}{\simeq} e_{x_1}$, we get that

$$\beta_{\overline{h}} \circ \beta_h([f]) = [h * \overline{h} * f * h * \overline{h}] = [f],$$

and hence $\beta_{\overline{h}} \circ \beta_h = \mathrm{Id}_{\pi_1(X,x_0)}$. Similarly, one shows that $\beta_h \circ \beta_{\overline{h}} = \mathrm{Id}_{\pi_1(X,x_1)}$, thus showing that β_h is a bijection with inverse map $\beta_{\overline{h}}$.

(2) We next need to verify that β_h is a group homomorphism. This is the case, because

$$\begin{aligned} \beta_h([f] \bullet [g]) &= \beta_h([f * g]) \\ &= [\overline{h} * f * g * h] \\ &= [\overline{h} * f * h * \overline{h} * g * h] \\ &= [\overline{h} * f * h] \bullet [\overline{h} * g * h] \\ &= \beta_h([f]) \bullet \beta_h([g]). \end{aligned}$$

The same argument works for $\beta_{\overline{h}}$, showing that we have an isomorphism. □

For path-connected spaces X, we will often simply write $\pi_1(X)$ if we only care about the structure of the group and not the basepoint.

6.2 The Fundamental Group and the Fundamental Theorem

In Lecture 5, we heuristically argued that $\pi_1(S^1) \cong \mathbb{Z}$, but didn't provide a formal proof. While intuitively clear, this result is not completely obvious to proof. As usual, we consider the circle $S^1 \subset \mathbb{C}$ as subset of the complex numbers, and define the loops with basepoint 1

$$\omega_n \colon I \to S^1, \quad \omega_n(s) = \exp(2\pi i n \cdot s)$$

for $n \in \mathbb{Z}$. Thus each ω_n goes around the circle $|n|$ times in counterclockwise (if $n > 0$) or clockwise (if $n < 0$) direction. In particular, $\omega_0 = e$ is the constant loop with basepoint 1. Note that $\omega_n = \omega_1^n$. The following theorem states the (apparently, but not, obvious) fact that every loop on S^1 with basepoint 1 is of this form.

Theorem 6.2. *The fundamental group $\pi_1(S^1, 1)$ is the infinite cyclic group generated by $[\omega_1]$, i.e.,*

$$\pi_1(S^1, 1) = \{[\omega_n] \mid n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

To prove this theorem, we have to make an excursion and discuss covering spaces. Before embarking on this theory, we'll show that this results allows to prove the Fundamental Theorem of Algebra [1]

Theorem 6.3. *Every non-constant, complex polynomial $p(z) \in \mathbb{C}[z]$ has at least one complex root, i.e., a $\lambda \in \mathbb{C}$ such that $p(\lambda) = 0$.*

Proof. The proof is by contradiction. Assume that there exists a polynomial

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

of degree $n \geq 1$ such that $p(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$ (we can without lack of generality assume $p(z)$ to be *monic*, meaning that the coefficient of z^n is 1). For every real number $r > 0$, such a polynomial gives rise to a loop g_r on S^1

$$g_r(s) = \frac{p(r\exp(2\pi i s))/p(r)}{|p(r\exp(2\pi i s)/p(r)|},$$

with basepoint $g_r(0) = g_r(1) = 1$. The strategy of the proof is two show, via two different homotopies, that

$$g_r \overset{\partial}{\simeq} e, \quad \text{and } g_r \overset{\partial}{\simeq} \omega_n.$$

Of course this should not be possible, giving the desired contradiction.

(1) Consider the homotopy $f_t = g_{tr}$. Then $f_1 = g_r$ and $f_0 = e$, and $[g_r] = [e]$.

[1] This celebrated result was first proved by J. Wood (1798) and C.F.Gauss (1799), but with subtle gaps. A first correct proof was given by J-R. Argand in 1806. Nowadays, countless algebraic, topological, geometric and analytic proofs are available.

(2) To show that g_r is homotopic to ω_n, the idea is to use p to construct a sequence of polynomial p_t that move continuously to z^n:

$$p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_{n-1}z + a_n).$$

If we can define, for some r, loops

$$\tilde{f}_t(s) = \frac{p_t(r\exp(2\pi is))/p_t(r)}{|p_t(r\exp(2\pi is)/p_t(r)|},$$

then $\tilde{f}_1 = g_r$ and $\tilde{f}_0 = \exp(2\pi is) = \omega_n$. To make sure that we can construct such $\tilde{f}_t$, we have to make sure that none of the quantities we are dividing by can be 0, or in other words, that the polynomials $p_t(z)$ have no roots with $|z| = r$. We will show that this is the case if r is big enough. More specifically, let r be such that

$$r > \max\{|a_1| + \cdots + |a_n|, 1\}.$$

Then for $z \in \mathbb{C}$ with $|z| = r$ we have

$$\begin{aligned} |z|^n &> (|a_1| + \cdots + |a_n|)|z|^{n-1} \\ &> |a_1||z|^{n-1} + |a_2||z|^{n-2} + \cdots + |a_{n-1}||z| + |a_n| \\ &\geq |a_1 z^{n-1} + \cdots + a_n|. \end{aligned}$$

In particular, for $t \in [0, 1]$, the polynomials p_t cannot have a root with $|z| = r$, as the absolute value of $|z|^n$ is always bigger than that of the rest of the terms. It follows that the homotopy $\tilde{f}_t$ is well defined. □

6.3 Covering spaces

Definition 6.4. A **covering** is a map $p\colon \tilde{X} \to X$ such that there exists an open cover $\{U_\alpha\}$ of X, such that for every α, the preimage is a disjoint union of open sets

$$p^{-1}(U_\alpha) = \bigsqcup_\beta V_\alpha^\beta,$$

and such that the restriction $p|_{V_\alpha^\beta}\colon V_\alpha^\beta \to U_\alpha$ is a homeomorphism.

Example 6.5. For $k \in \mathbb{Z}$, the maps $p_k\colon S^1 \to S^1$, $z \mapsto z^k$ are covering maps. The map $p_\infty\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi ikt)$ is a covering map.