Lecture 7

In this lecture we introduce and study covering spaces in some detail.

7.1 Covering spaces

Definition 7.1. A **covering** is a map $p\colon \tilde{X} \to X$ such that there exists an open cover $\{U_\alpha\}$ of X, such that for every α, the preimage is a disjoint union of open sets

$$p^{-1}(U_\alpha) = \bigsqcup_\beta V_\alpha^\beta,$$

and such that the restriction $p|_{V_\alpha^\beta}\colon V_\alpha^\beta \to U_\alpha$ is a homeomorphism.

Example 7.2. For $k \in \mathbb{Z}$, the maps $p_k\colon S^1 \to S^1$, $z \mapsto z^k$ are covering maps. The preimage $p^{-1}(z)$ of any point $z = \exp(2\pi i t) \in S^1$ consists of precisely k distinct points, namely $\exp(2\pi i(t+j)/k)$ for $j \in \{0, \ldots, k-1\}$. For $z = 1$, these are precisely the k-th complex roots of unity.

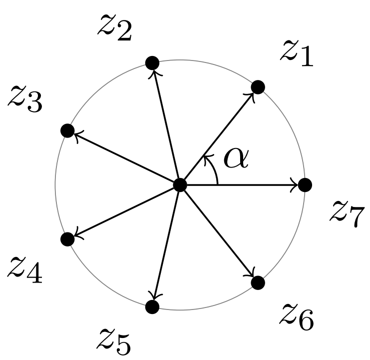

Figure 7.1: The preimage $p_7^{-1}(1)$.

Example 7.3. The map $p_\infty\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$ is a covering map. The preimage $p_\infty^{-1}(z)$ consists of ∞ many points.

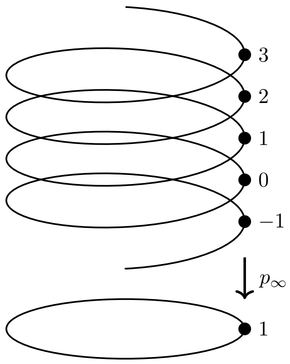

Figure 7.2: The preimage $p_\infty^{-1}(1)$.

Definition 7.4. A covering $p\colon \tilde{X} \to X$ is called an n**-fold covering** if for all $x \in X$, $p^{-1}(x)$ consists of precisely n points.

Definition 7.5. Two coverings $p\colon Y \to X$ and $q\colon Z \to X$ are called **isomorphic**, if there exists a homeomorphism $h\colon Y \to Z$ such that $p = q \circ h$.

It is common to visualize concepts such as the isomorphism of coverings via **commutative diagrams** such as the following.

$$\begin{array}{ccc} Y & \xrightarrow[\cong]{h} & Y \\ & {}_{p}\searrow \quad \swarrow_{q} & \\ & X & \end{array}$$

The requirement is, that all compositions in such a diagram should coincide.

Example 7.6. The coverings $p_2\colon S^1 \to S^1$ and $p_{-2}\colon S^1 \to S^1$ are isomorphic: the homeomorphism $h\colon S^1 \to S^1$, $h(z) = -z$, satisfies $p_{-2} = p_2 \circ h$.

Example 7.7. The coverings $p_2\colon S^1 \to S^1$ and $p_3\colon S^1 \to S^1$ are not isomorphic: one is a 2-fold covering and the other is a 3-fold covering.

Definition 7.8. Let $p\colon \tilde{X} \to X$ be a covering. A **Deck transformation** is a homeomorphism $\tau\colon \tilde{X} \to \tilde{X}$ such that $p \circ \tau = p$, i.e., τ gives rise to an isomorphism of a covering to itself. The set of all Deck transformations of a cover is called $\mathrm{Deck}(p)$.

Exercise 7.9. Show that $(\mathrm{Deck}(p), \circ)$, where $\circ$ is the composition of maps, is a group.

Example 7.10. The map $\tau\colon S^1 \to S^1$, $z \mapsto -z$ gives a Deck transformation for the cover $p_2\colon S^1 \to S^1$.

Exercise 7.11. Show that for $m \in \mathbb{Z}$, the maps $\tau_m\colon \mathbb{R} \to \mathbb{R}$, $t \mapsto t + m$, give a Deck transformation for the cover $p_\infty\colon \mathbb{R} \to S^1$. Conclude that $\mathrm{Deck}(p_\infty) \cong \mathbb{Z}$.

Definition 7.12. Let $p\colon \tilde{X} \to X$ be a covering and $f\colon Y \to X$ any map. A **lift** of f is a map $\tilde{f}\colon Y \to \tilde{X}$ such that $f = p \circ \tilde{f}$.

The requirement $f = p \circ \tilde{f}$ is often visualized using the following diagram.

$$\begin{array}{ccc} & & \tilde{X} \\ & \overset{\tilde{f}}{\nearrow} & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[f]{} & X \end{array}$$

Example 7.13. Consider a loop $f\colon I \to S^1$, $t \mapsto \exp(2\pi i n t)$ and the covering p_∞. Then the map $\tilde{f}\colon I \to \mathbb{R}$, $t \mapsto nt$, is a lift of f.