Lecture 8

In this lecture we will construct a homomorphism from $\mathbb{Z}$ to the fundamental group $\pi_1(S^1, 1)$. To construct this homomorphism, we need to study properties of liftings.

8.1 Properties of Liftings

Given a covering $p\colon \tilde{X} \to X$, recall that a lift of $f\colon Y \to X$ was a map $\tilde{f}\colon Y \to \tilde{X}$ such that $f = p \circ \tilde{f}$,

$$\begin{array}{ccc} & & \tilde{X} \\ & \overset{\tilde{f}}{\nearrow} & \big\downarrow p \\ Y & \xrightarrow[f]{} & X \end{array}$$

Lemma 8.1. *Let $p\colon \tilde{X} \to X$ be a cover and $\tilde{f}, \tilde{g}\colon Y \to \tilde{X}$ maps. Then:*

(1) $\tilde{f}$ is a lift of $p \circ \tilde{f}$;

(2) If $\tilde{f} \simeq \tilde{g}$, then $p \circ \tilde{f} \simeq p \circ \tilde{g}$ ("Homotopies descend");

(3) If $\alpha, \beta\colon I \to \tilde{X}$ are paths with $\alpha(1) = \beta(0)$, then $p \circ (\alpha * \beta) = (p \circ \alpha) * (p \circ \beta)$ ("Paths descend").

Proof. Property (1) is obvious from the definition of a lift. For property (2), observe that any homotopy $\tilde{f}_t$ from $\tilde{f}$ to $\tilde{g}$ gives rise to a homotopy $p \circ \tilde{f}_t$ from $p \circ \tilde{f}$ to $p \circ \tilde{g}$. For property (3), note that

$$p \circ (\alpha * \beta)(t) = \begin{cases} p \circ \alpha(2t) & t \le 1/2 \\ p \circ \beta(2t-1) & t \ge 1/2 \end{cases},$$

which is the same as $(p \circ \alpha) * (p \circ \beta)(t)$. □

8.2 Loops on S^1

Recall the map

$$\begin{aligned} \omega_n\colon I &\to S^1 \\ t &\mapsto \exp(2\pi i n t). \end{aligned}$$

The map $\tilde{\omega}_n\colon I \to \mathbb{R}$, $t\colon t+n$ is clearly a lift of ω_n, i.e., it satisfies

$$\omega_n = p \circ \tilde{\omega}_n. \tag{A}$$

Consider also the Deck transformation $\tau_n\colon \mathbb{R} \to \mathbb{R}$, $t \mapsto t+n$, and the composition $\tilde{\omega}_m * (\tau_m \circ \tilde{\omega}_n)$. This composition is a path in $\mathbb{R}$ from 0 to $m+n$, and therefore homotopic to $\tilde{\omega}_{m+n}$,

$$\tilde{\omega}_{n+m} \simeq \tilde{\omega}_m * (\tau_m \circ \tilde{\omega}_n), \tag{B}$$

as can be seen using the straight-line homotopy $f_t = (1-t)\tilde{\omega}_{n+m} + t\tilde{\omega}_m * (\tau_m \circ \tilde{\omega}_n)$.

We now have everything in place to construct a homomorphism of $\mathbb{Z}$ to the fundamental group of the circle. Define the map

$$\begin{aligned} \Phi\colon \mathbb{Z} &\to \pi_1(S^1, 1) \\ n &\mapsto [\omega_n]. \end{aligned}$$

Lemma 8.2. *The map $\mathbb{Z}$ is a group homomorphism.*

Proof. We need to show that $[\omega_{m+n}] = [\omega_m] \bullet [\omega_n]$:

$$\begin{aligned} \Phi(m+n) &= [\omega_{m+n}] \\ &\overset{\text{(A)}}{=} [p \circ \tilde{\omega}_{m+n}] \\ &\overset{\text{Lemma (8.1)(2)+(B)}}{=} [p \circ (\tilde{\omega}_m * (\tau_m \circ \tilde{\omega}_n))] \\ &\overset{\text{Lemma (8.1)(3)}}{=} [p \circ \tilde{\omega}_m * p \circ \tau_m \circ \tilde{\omega}_n] \\ &= [p \circ \tilde{\omega}_m] \bullet [p \circ \tau_m \circ \tilde{\omega}_n] \\ &\overset{\tau_m \in \mathrm{Deck}(p)}{=} [p \circ \tilde{\omega}_m] \bullet [p \circ \tilde{\omega}_n] \\ &\overset{\text{(A)}}{=} [\omega_m] \bullet [\omega_n] \\ &= \Phi(m) \bullet \Phi(n). \end{aligned}$$

□

The philosophy of the proof is that we have shown something about loops on S^1 by considering a cover of S^1, $p_\infty\colon \mathbb{R} \to S^1$, and working in $\mathbb{R}$. Things are very simple in $\mathbb{R}$: the crucial property (B) is easy to prove and shows that the composition of two lifts $\tilde{\omega}_m$ and $\tilde{\omega}_n$ (up to a reparametrization given by the Deck transformation τ_m) is homotopic to the lift $\tilde{\omega}_{m+n}$. Using the property that “homotopies descend” and “paths descend”, we can transfer things proved “upstairs” to “downstairs”. We will see in the next lecture that the map Φ is bijective, thus giving an isomorphism.