Lecture 9

In the previous lecture we looked at the map

$$\begin{aligned} \Phi\colon \mathbb{Z} &\to \pi_1(S^1, 1) \\ n &\mapsto [\omega_n], \end{aligned}$$

where $\omega_n(t) = \exp(2\pi i t)$ is the "n times around" loop. We saw that this map is a group homomorphism, meaning that $[\omega_{n+n}] = [\omega_n] \bullet [\omega_m]$. The way this was shown was to consider the covering $p_\infty\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$, and *lifted* paths $\tilde{\omega}_{n+n}$, $\tilde{\omega}_n$, and $\tilde{\omega}_m$ from I to $\mathbb{R}$ such that $p \circ \tilde{\omega}_n = \omega_n$. It was relatively easy to establish a homotopy between $\tilde{\omega}_{n+m}$ and the concatenation of $\tilde{\omega}_n$ with a shifted version of $\tilde{\omega}_m$, and this homotopy descends to a homotopy $\omega_{n+n} \simeq \omega_n * \omega_m$.

What the proof does not show yet, is that this is bijective: we don't know whether Φ hits all the elements of $\pi_1(S^1, 1)$, and whether two distinct $n \neq m$ give rise to distinct classes $[\omega_n]$ and $[\omega_m]$. The latter is equivalent to the important statement that for all $m \in \mathbb{Z}$, $\omega_m \simeq e \Leftrightarrow m = 0$ (where e is the constant loop at 1). This statement is non-trivial, and relies on the fact that homotopies in the base space of a covering "lift" to homotopies in the covering space.

9.1 The homotopy lifting property

Recall the convention that for a homotopy $F\colon Y \times I \to X$ we write $f_t(y) = F(y, t)$.

Definition 9.1. Let $p\colon Z \to X$ be a map. Then p has the **Homotopy Lifting Property** (HLP) if given a homotopy $F\colon Y \times I \to X$ and a *lift* $g\colon Y \times \{0\} \to Z$ of f_0, so that $f_0 = p \circ g$, there exists a *unique* homotopy $\tilde{F}\colon Y \times I \to Z$ such that

(i) $\tilde{f}_0 = g$;

(ii) $p \circ \tilde{F} = F$.

In terms of diagrams,

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{\ g\ } & Z \\ {\scriptstyle \iota}\big\downarrow & \overset{\tilde{F}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ Y \times I & \xrightarrow{\ F\ } & X \end{array}$$

Recall that we use the notation $Y \hookrightarrow X$ to denote the *inclusion map* of a subspace. The diagram is required to commute, i.e., all compositions coincide (for example, $p \circ g = F \circ \iota$). The dashed line means that we require the existence of a map $\tilde{F}$ making the diagram commute. Note that Condition (i) above says that the upper triangle in the diagram commutes ($\tilde{f}_0 = \tilde{F} \circ \iota = g_0$) and Condition (ii) is equivalent to the commutativity of the lower triangle.

An important special case is the **Path Lifting Property**, or homotopy lifting property for paths.

Definition 9.2. Let $p\colon Z \to X$ be a map. Then p satisfies the homotopy lifting property for paths, or the Path Lifting Property (PLT), if for any path $f\colon I \to X$ with $f(0) = x_0$ and $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique path $\tilde{f}\colon I \to Z$ with $\tilde{f}(0) = \tilde{x}_0$ and $p \circ \tilde{f} = f$.

Note that the path lifting property is a special case of the HLP with $Y = \{\mathrm{pt}\}$. In this case, the homotopy F is simply a path

$$F\colon \{\mathrm{pt}\} \times I \to X,$$

$f_0\colon \{\mathrm{pt}\} \times \{0\} \to X$ is simply a point $x_0 \in X$, and $g\colon \{\mathrm{pt}\} \times \{0\}$ is simply a point $\tilde{x}_0 \in Z$. Denoting $f(t) = F(\mathrm{pt}, t)$, we recover Definition 9.2.

Proposition 9.3. *A covering map $p\colon \tilde{X} \to X$ satisfies the Homotopy Lifting Property.*

Example 9.4. Consider the covering $p_\infty\colon \mathbb{R} \to S^1$. If $z_0 = \exp(2\pi i t_0) \in S^1$, then $p_\infty^{-1}(z_0) = \{t_0 + m \mid m \in \mathbb{Z}\}$. Consider the path $\omega\colon I \to S^1$ given by $\omega(t) = \exp(2\pi i(t + t_0))$. Then for any $m \in \mathbb{Z}$ there exists a unique path $\tilde{\omega}\colon I \to \mathbb{R}$, given by $\tilde{\omega}(t) = t_0 + m + t$ such that $p_\infty \circ \tilde{\omega} = \omega$. The choice of point in the preimage $p_\infty^{-1}(z_0)$ corresponds to the choosing a branch of the complex logarithm.

9.2 The fundamental group of the circle

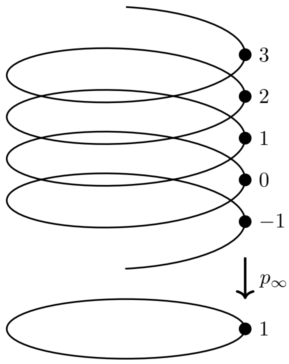
Theorem 9.5. *The map $\Phi\colon \mathbb{Z} \to \pi_1(S^1, 1)$, $n \mapsto [\omega_n]$, is a group isomorphism.*

Proof. We already saw that Φ is a homomorphism, and only need to show that it is bijective.

We first show that the map is surjective: if $[\alpha] \in \pi_1(S^1, 1)$ then there exists $n \in \mathbb{Z}$ with $[\alpha] = [\omega_n]$. Consider again the cover $p = p_\infty\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$. Given a loop α, apply the *path lifting property* to see that there exists a unique lift $\tilde{\alpha}\colon I \to \mathbb{R}$ such that

(i) $p \circ \tilde{\alpha} = \alpha$;

(ii) $\tilde{\alpha}(0) = 0$.



Since $\alpha(1) = 1$ (α is a loop starting and ending at $1 \in S^1$) and $p \circ \tilde{\alpha} = \alpha$, we have $\tilde{\alpha}(1) \in p^{-1}(1) = \mathbb{Z}$, say $\tilde{\alpha}(1) = n$.

Therefore $\tilde{\alpha} \overset{\partial}{\simeq} \tilde{\omega}_n$, since both are paths from 0 to n in $\mathbb{R}$, with a homotopy given by the straight-line homotopy $f_t = (1-t)\tilde{\alpha} + t\tilde{\omega}_n$. Since homotopies descend, we get

$$\alpha = p \circ \tilde{\alpha} \overset{\partial}{\simeq} p \circ \tilde{\omega} = \omega_n,$$

which implies $[\alpha] = [\omega_n]$.

To show injectivity, assume that $\Phi(n) = [\omega_n] = [e]$, i.e., $\omega_n \overset{\partial}{\simeq} e$, the constant loop. This means that there is a homotopy of loops

$$F\colon I \times I \to S^1$$

with $f_0 = \omega_n$, $f_1 = e$, and $f_t(0) = f_t(1) = 1$ for all t. Define $g\colon I \times \{0\} \to \mathbb{R}$ by setting $g(s,0) = \tilde{\omega}_n$. By Proposition 9.3, the covering p satisfies the HLP, and we therefore have a homotopy $\tilde{F}\colon I \times I \to \mathbb{R}$ such that $\tilde{f}_0 = g$ and $p \circ \tilde{F} = F$. The other end of the homotopy, $\tilde{f}_1$, satisfies $p \circ \tilde{f}_1 = e$, the constant loop, and therefore $\tilde{f}_1 \in \mathbb{Z}$. Since any path $I \to \mathbb{Z}$ must be constant, $\tilde{f}_1 = n$. On the other hand, also $\tilde{f}_t(0) \in \mathbb{Z}$, since any $p \circ \tilde{f}_t(0) = f_t(0) = 1$. It follows that $\tilde{f}_1(0) = \tilde{f}_0(0) = 0$, and since we saw that $\tilde{f}_1(s)$ is constant, $\tilde{f}_1(s) = n = 0$. This completes the proof. □