Example Sheet 1

Please let me (Martin) know if any of the problems are unclear or have typos. Please turn in a solution to one of Exercises (1.2), (1.3), or (1.7) by 14:00 on 11/10/2018, to the dropoff box in front of the undergraduate office. If you collaborate with other students, please include their names.

(1.1) Suppose that X is a topological space. Define $\mathrm{Homeo}(X)$ to be the set of homeomorphisms $f\colon X \to X$. Show that $\mathrm{Homeo}(X)$ is a group if we take the group operation to be function composition.

Give an example of a space X where $\mathrm{Homeo}(X)$ is the trivial group.

(1.2) Show that the relation $X \cong Y$ of being homeomorphic is an equivalence relation on topological spaces. Now consider the capital letters of the alphabet A, B, C, ... in a sans serif font. Each of these gives a graph in the plane. Sort these into homeomorphism classes. (The partition may depend on the font! In particular, K can be tricky.)

(1.3) We equip $[0, 1) \subset \mathbb{R}$ and $S^1 \subset \mathbb{C}$ with their usual subspace topologies. Consider the map $p\colon [0, 1) \to S^1$ given by $p(t) = \exp(2\pi i t)$. Show that p is a continuous bijection. Show that p is not a homeomorphism.

(1.4) We equip $[0, 1] \subset \mathbb{R}$ and $S^1 \subset \mathbb{C}$ with their usual subspace topologies. Show that the quotient space

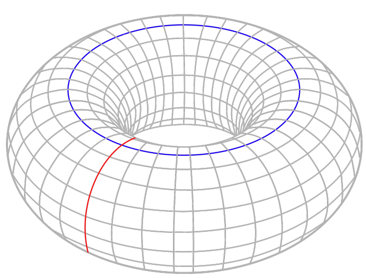
$$X = {}^{[0,1]}\!/\!_{0\sim 1}$$

is homeomorphic to S^1.

(1.5) The embedded torus is parametrized as follows:

$$X = \{((b+a\cos(\theta))\sin(\varphi), (b+a\cos(\theta))\cos(\varphi), a\sin(\theta)) \in \mathbb{R}^3 \mid \theta \in [0, 2\pi), \varphi \in [0, 2\pi)\}$$

for some $0 < a < b$. Show that $X \cong S^1 \times S^1$ (hence the name torus for $\mathbb{T}^2 = S^1 \times S^1$).



(1.6) For three of the following pairs (X, Y) show that X is not homeomorphic to Y.

- The graph X and the graph Y.
- $(0, 1)$ and $[0, 1]$: the open and closed intervals.
- S^1 and $[0, 1]$: the circle and the closed interval.
- S^1 and S^2: the circle and the sphere.
- $\mathbb{R}^1$ and $\mathbb{R}^2$: the line and the plane.
- $\mathbb{R}^2$ and $\mathbb{R}^3$: the plane and three-space (harder).
- S^2 and $\mathbb{T}^2 = S^1 \times S^1$: the sphere and the torus (harder).

(1.7) Suppose X and Y are topological spaces. We call a function $f\colon X \to Y$ an *embedding* if f is a homeomorphism from X to $f(X)$, equipped with the subspace topology. Give an example of a space X that does not embed in $\mathbb{R}^n$, for any n.