Example Sheet 2

Please let me (Martin) know if any of the problems are unclear or have typos. Please turn in a solution to one of Exercises (2.1), (2.4) or (2.6) by 14:00 on 18/10/2018, to the dropoff box in front of the undergraduate office. If you collaborate with other students, please include their names.

(2.1) Suppose that $F\colon X \times I \to Y$ is continuous. For each $t \in I$ define $f_t\colon X \to Y$ by $f_t(x) = F(x, t)$. Prove that f_t is continuous.

(2.2) Show that the relation $f \simeq g$ of being homotopic is an equivalence relation on maps.

(2.3) Show that $f \simeq g$ implies $h \circ f \simeq h \circ g$ (assuming that all compositions make sense). Show that the relation $X \simeq Y$ of being homotopy equivalent is an equivalence relation on topological spaces.

(2.4) Show that $\mathbb{R}^n - \{0\} \cong S^{n-1} \times \mathbb{R} \simeq S^{n-1}$. That is, the first pair of spaces are homeomorphic while the second pair are homotopy equivalent. Use this to prove that $\mathbb{R}^n - \{0\} \simeq S^{n-1}$.

(2.5) Fix $m, n \in \mathbb{N}$ so that $0 < n < m$. We embed $\mathbb{R}^n$ into $\mathbb{R}^m$ by taking $(x_1, \ldots, x_n) \in \mathbb{R}^n$ to $(x_1, \ldots, x_n, 0, \ldots, 0) \in \mathbb{R}^m$. Show that $\mathbb{R}^m - \mathbb{R}^n \simeq S^{m-n-1}$.

(2.6) [Medium.] Consider the capital letters of the alphabet A, B, C, ... in a sans serif font. Each of these gives a graph in the plane. Sort these into homotopy equivalence classes. Clearly state any unproven assumptions that you rely on.

(2.7) [Medium.] Show that the eyeglasses graph E and the theta graph T (both shown in Figure 1) are homotopy equivalent.

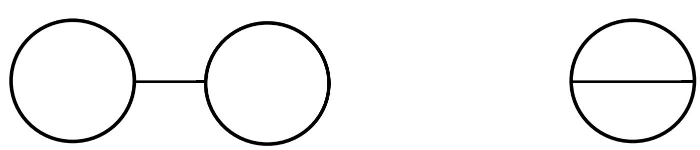

Figure 1: Left: the eyeglasses graph. Right: the theta graph.

(2.8) Define $\omega_n\colon I \to S^1$ by $\omega_n(t) = \exp(2\pi i n t)$. Show that the concatenation $\omega_p * \omega_q$ is homotopic, rel endpoints, to ω_{p+q}. (First work out the special case of $p = 2$ and $q = 1$. Then do the general case.)