Example Sheet 3

Please let me (Martin) know if any of the problems are unclear or have typos. Please turn in a solution to one of Exercises (3.1) and (3.3) by 14:00 on 25/10/2018, to the dropoff box in front of the undergraduate office. If you collaborate with other students, please include their names.

For the first three problems the paths $f, g, h\colon I \to X$ are loops based at the point $x_0 \in X$. The path $e\colon I \to X$ is the constant loop, also based at x_0.

(3.1) Give an explicit parameterization of the loop $e * f$. Show, by giving a picture in $I \times I$, a picture in X, and an explicit homotopy, that $e * f$ is homotopic (preserving endpoints) to f.

(3.2) Let X be a path-connected space. We have seen that for any two x_0, x_1, the fundamental groups of (X, x_0) and (X, x_1) are isomorphic: $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. The isomorphism is given by a map $\beta_h\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$, mapping $[f]$ to $[\overline{h} * f * h]$, where h is a path from x_0 to x_1 and $\overline{h}$ the inverse path. Show that the fundamental group $\pi_1(X, x_0)$ is abelian if and only if all the basepoint-change homomorphisms β_h depend only on the endpoints (and not the particular path taken).

(3.3) Give explicit parameterizations of the loops $p = (f * g) * h$ and $q = f * (g * h)$. Show, by giving a picture in $I \times I$, a picture in X, and an explicit homotopy, that p and q are homotopic (preserving endpoints).

(3.4) A *graph* consists of a collection of *vertices* V and *edges* E joining them. Formally, V is just a set and E consists of subsets $e = \{x, y\}$ with $x, y \in V$. Assign to each edge $e \in E$ an interval $I_e = [0, 1]$. A (topological) graph is formed by assigning to each edge a (not necessarily distinct) pair of vertices via a map $\varphi_e\colon \partial I_e \to V$, where $\partial I_e = \{0, 1\}$, and taking the quotient of the disjoint union of edges and vertices, $V \sqcup \bigsqcup_{e \in E} I_e / \sim$, where $x \sim \varphi_e(x)$ for every $x \in \partial I_e$.

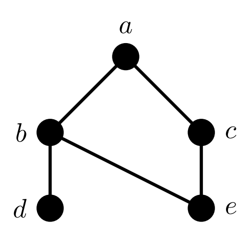

Figure 1: A graph.

- Let $X \subset \mathbb{R}^3$ be the union of the coordinate axes. Show that $\mathbb{R}^3 - X$ is homotopy equivalent to a graph. Which graph?
- Let $X \subset \mathbb{R}^4$ be the union of the xy–plane and the zw–plane. Show that $\mathbb{R}^4 - X$ is homotopy equivalent to a surface. Which surface?

(3.5) Suppose that $p\colon \tilde{X} \to X$ is a covering map. Recall the definition of $\mathrm{Deck}(p)$ and prove it is a group (using composition of functions as the binary operation).

(3.6) Show that the map $p\colon \mathbb{R} \to S^1$ defined by $p(t) = \exp(2\pi i t)$ is a covering map. Give an informal proof that $\mathrm{Deck}(p) \cong \mathbb{Z}$.