

Example Sheet 4

Please let me (Martin) know if any of the problems are unclear or have typos. Please turn in a solution to one of Exercises (4.1) or (4.2) by 14:00 on 1/11/2018, to the dropoff box in front of the undergraduate office. If you collaborate with other students, please include their names.

(4.1) Let Q_n be the space of n points, equipped with the discrete topology. Find a two-fold cover $p: Q_4 \rightarrow Q_2$ and compute its deck group.

(4.2) Define \mathbb{RP}^2 to be the space of lines through the origin in \mathbb{R}^3 . Formally, \mathbb{RP}^2 is the quotient space obtained from $\mathbb{R}^3 - \{0\}$ by identifying two vectors u, v if and only if there is a non-zero real λ so that $u = \lambda v$. Construct a two-fold covering map $p: S^2 \rightarrow \mathbb{RP}^2$.

(4.3) Let F be the *figure-eight graph*: the graph with one vertex and two edges. List all connected two- and three-fold covers of F , up to isomorphism. Sketch an argument showing that your lists are complete and irredundant.

(4.4) Describe the Deck group for each of the covers of F given in Exercise (4.3).

(4.5) With notation as set in class: check the following claims, needed in the proof that Φ is a homomorphism.

- Show that $\tilde{\omega}_{m+n} \stackrel{\partial}{\simeq} \tilde{\omega}_m * (\tau_m \circ \tilde{\omega}_n)$.
- Suppose that $\alpha \stackrel{\partial}{\simeq} \beta$ are paths in \mathbb{R} . Show that $p \circ \alpha \stackrel{\partial}{\simeq} p \circ \beta$ as paths in S^1 .
- Suppose that α and β are paths in \mathbb{R} with $\alpha(1) = \beta(0)$. Show that $p \circ (\alpha * \beta) = (p \circ \alpha) * (p \circ \beta)$.

(4.6) [Problem 6, page 38, Hatcher] We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, 1) \rightarrow (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \rightarrow X$, with no conditions on basepoints. Thus there is a natural map $\Phi: \pi_1(X, x_0) \rightarrow [S^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X, x_0)$, when X is path-connected.

(4.7) Suppose that $p: \tilde{X} \rightarrow X$ is a covering map. Suppose that $x_0 \in X$ and $e: I \rightarrow X$ is the constant path at x_0 . Give a direct proof (without using Proposition 10.1) that any lift \tilde{e} of e is again a constant path.

(4.8) [Medium] Boil the proof of Proposition 10.1 all the way down to prove that covering maps have the (unique) *path lifting property*: suppose that $p: \tilde{X} \rightarrow X$ is a covering map, suppose that $f: I \rightarrow X$ is a path, and set $x_0 = f(0)$. Suppose that $\tilde{x}_0 \in \tilde{X}$ is sent to x_0 by p . Then there is a unique lift \tilde{f} of f so that $\tilde{f}(0) = \tilde{x}_0$.