Week 5-8 Feedback (Part I)

These notes provide feedback on some common questions arising in weeks 5-8.

1 Lebesgue covering Lemma

Let \((X, d)\) be a metric space. Recall that the \textbf{diameter} of a set \(A \subset X\) is defined as \(\text{diam}(A) = \sup_{x,y \in A} d(x, y)\), and the distance to a set is defined as \(d(x, C) = \inf\{y \in C : d(x, y)\}\). In the lecture we sometimes made use of a special case of the following Lemma. While we did not need this generality, it is worth including it for completeness (you may have seen something similar in Metric Spaces).

\textbf{Lemma 1.1.} Let \((X, d)\) be a compact metric space \(X = \bigcup_{\alpha} U_\alpha\) an open cover. Then there exists \(\varepsilon > 0\) such that for every \(A \subset X\) with \(\text{diam}(A) < \varepsilon\), \(A \subset U_\alpha\) for some \(\alpha\).

\textit{Proof.} By compactness, there exists a finite subcover \(\{U_1, \ldots, U_m\}\). Assume that \(U_i \neq X\) for some \(i\) (otherwise we are done) and set \(C_i = X - U_i\) for \(i \in \{1, \ldots, m\}\). Assume that for every \(n \geq 1\) there exists a point \(x_n\) such that \(d(x_n, C_i) < 1/n\) for all \(i\). Since \(X\) is compact and since compact metric spaces are sequentially compact, the sequence \(\{x_n\}\) has a convergent subsequence \(\{x_{n_i}\}\), with limit \(x\). This \(x\) is a limit point of the closed set \(\bigcap_i C_i\), and therefore \(x \in \bigcap_i C_i\). On the other hand, \(\bigcap_i C_i = X - \bigcup_i U_i = \emptyset\), a contradiction. It follows that there exists \(\varepsilon > 0\) such that for all \(x \in X\), there is a \(C_i\) with \(\text{dist}(x, C_i) \geq \varepsilon\). In other words, every open ball \(B_x\) of radius \(\varepsilon\) is contained in some \(U_i\), and hence also every set with diameter \(< \varepsilon\). \(\square\)

See Wikipedia, or Topology: A first course by J. Munkres, for a different proof. In our applications, we usually have a curve \(\gamma : I \rightarrow X\) and a cover \(X = \bigcup_{\alpha} U_\alpha\), and then deduce that we can subdivide the interval \(I\) by finding \(t_0 < t_1 < \cdots < t_n\) such that each subinterval satisfies \(\gamma([t_i, t_{i+1}]) \subset U_\alpha\) for some \(\alpha\). While the existence of such a subdivision follows from the above lemma, it can also be shown more directly.

2 An explicit homotopy

In the lecture we used the fact that \(\mathbb{R}^2\) without two points (say, \((-0.5, 0)\) and \((0.5, 0)\)) is homotopy equivalent to the figure-eight \(S^1 \vee S^1\):

![Figure-eight](image)

As mentioned early in the lecture, in topology one rarely describes such homotopy equivalences explicitly using formulas, once one is convinced that it is in principle possible. In this particular case, a hint to the fact that this is possible is given by the similar structure: the “outer space” in \(\mathbb{R}^2\) can be retracted to a circle containing the two
points, and the points can be expanded, so that the resulting spaces look very similar (with one of them having "thicker walls"):

As also mentioned in one of the early problem sheets, instead of finding an explicit deformation retract or homotopy from one space to the other, one can alternatively use a third space as a "proxy". In our case, it would seem convenient to use the following graph:

A retraction from \( X = \mathbb{R}^2 - \{(-0.5, 0), (0.5, 0)\} \) to this graph (let’s call it \( \Theta \)) is easily described by mapping any point \( x \) outside the circle to \( x/\|x\| \), any point on the left-side to the intersection of the line segment emanating from \((-0.5, 0)\) through \( x \) with \( \Theta \), and the same on the right half:

A deformation retract is then achieved by attaching a straight-line homotopy to this retraction, i.e., if \( r(x) \) is the retraction, then we define

\[
F(x, t) = (1 - t)x + tr(x).
\]

The graph \( \Theta \) then deformation retracts to the figure-eight in the obvious way, by squashing the line in the middle towards a point (if still unsure, try to find equations for all the transformations above, in the same way that we found equations, for example, for describing the homotopy from \( D^n/S^{n-1} \) to \( S^n \) in an earlier sheet).
3 On the order of taking products and quotients

Suppose that \( X = \tilde{X}/\sim \) is a quotient space, and we would like to construct a homotopy

\[ F: X \times I \to Y. \]

A canonical approach is to first consider a homotopy \( \tilde{F}: \tilde{X} \times I \to Y \) and then “factor” it through the quotient,

\[ \tilde{X} \times I \quad \xrightarrow{q \times \text{Id}} \quad X \times I \quad \xrightarrow{F} \quad Y \]

We have seen cases of this before, for example when showing that the torus without a point deformation retracts to the figure-eight. We would consider the deformation of a square without a point, \( X = I^2 - \{\text{pt}\} \), to its boundary “frame” and then apply the torus identifications (pasting the left to the right boundary, and the upper to the lower) to get the figure-eight.

The problem with this type of construction is that it is not a priori obvious that the function \( F \) we arrive at is even continuous! The reason is that taking the product of a quotient is not the same as taking the quotient of a product, and hence we cannot be sure that \( F^{-1}(U) \) is open, even if \( \tilde{F}^{-1}(U) \) is open. In the situation above, however, this approach does work. To formalize this, we need the notion of an identification map: this is a surjective map \( f: X \to Y \) such that \( U \subset Y \) is open if and only if \( f^{-1}(U) \) is open (\( Y \) has the quotient topology). For example, the quotient map \( q: X \to X/\sim \) onto a quotient space is an identification map. Identification maps satisfy the following property (which you should try to prove).

**Proposition 3.1.** A map \( f: X \to Y \) is an identification map if and only if for every space \( Z \) and every function \( g: Y \to Z \), \( g \circ f \) is continuous if and only if \( g \) is continuous.

\[ X \quad \xrightarrow{f} \quad Y \quad \xrightarrow{g} \quad Z \quad \xrightarrow{g \circ f} \quad Z \]

The following proposition solves all our problems (well, maybe not all).

**Proposition 3.2.** If \( f: X \to Y \) is an identification map, then \( f: X \times I \to Y \times I \) is an identification map.

**Proof.** We use Proposition ???. Let \( g: Y \times I \to Z \) be a function and consider the composition \( h = g \circ (f \times \text{Id}): X \times I \to Z \). Clearly, if \( g \) is continuous, then so is \( h \). Assume now that \( h \) is continuous. We would like to show that \( g \) is continuous.

Let \( U \subset Z \) be an open set and let \( (x, t) \in h^{-1}(U) \). Since \( h \) is continuous, \( h^{-1}(U) \) is open in \( X \times I \), and by the property of the product topology there exists an interval \( [s, t] \subset I \) such that \( h(\{x\} \times [s, t]) \subset U \). Define

\[ V = \{y \in Y : g(\{y\} \times [s, t]) \subset U\}. \]
Since $f(x) \in V$, this set is not empty and the preimage under $f$ is given by

$$f^{-1}(V) = \{x \in X : h(\{x\} \times [s,t]) \subset U\}.$$ 

Since $f$ was assumed to be an identification map, $V$ is open if and only if $f^{-1}(V)$ is open. So we will show that $f^{-1}(V)$ is open.

Note that if $x \in f^{-1}(V)$, then for all $s' \in [s,t]$,

$$(x, s') \notin C := (X \times [s,t]) \cap h^{-1}(Z - U).$$

Since $C$ is closed, this means that for each $s' \in [s,t]$ there exists open sets $U_{s'}$ and $V_{s'}$ with $x \in U_{s'}$ such that

$$(U_{s'} \times V_{s'}) \cap C = \emptyset.$$ 

By compactness, there exist finitely many open sets $U_i, V_i$ among those, such that

$$\bigcup_i V_i = [s,t].$$

Then

$$\left(\bigcap_i U_i \times [s,t]\right) \cap C = \left(\bigcap_i U_i \times \bigcup_i V_i\right) \cap C = \emptyset.$$ 

It follows that $x \in \bigcap_i U_i \subset f^{-1}(V)$, and since $\bigcap_i U_i$ is open, that $f^{-1}(V)$ is open. \qed

As an application of this result, we can show that a cone over a topological space is contractible (Exercise 7.3) or formally show that the torus without a point deformation retracts to a figure-eight (Exercise 6.1). The result holds for any locally compact space instead of $I$, see I.13.19 in Geometry and Topology by G. Bredon.

4 To be continued...

In the second part of this document, I will discuss Problem (6.5) in great detail, the computation of the fundamental group of projective space, and the solution of some assorted problems.