Week 5-8 Feedback (Part I)

These notes provide feedback on some common questions arising in weeks 5-8.

1 Lebesgue covering Lemma

Let (X, d) be a metric space. Recall that the **diameter** of a set $A \subset X$ is defined as $\operatorname{diam}(A) = \sup_{x,y \in A} d(x, y)$, and the distance to a set is defined as $d(x, C) = \inf\{y \in C : d(x, y)\}$. In the lecture we sometimes made use of a special case of the following Lemma. While we did not need this generality, it is worth including it for completeness (you may have seen something similar in Metric Spaces).

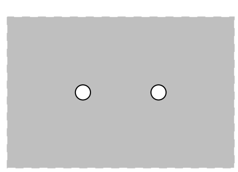
Lemma 1.1. *Let (X, d) be a compact metric space $X = \bigcup_\alpha U_\alpha$ an open cover. Then there exists $\varepsilon > 0$ such that for every $A \subset X$ with $\operatorname{diam}(A) < \varepsilon$, $A \subset U_\alpha$ for some α.*

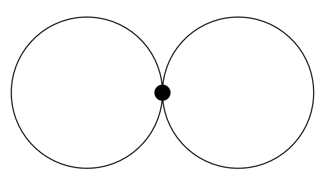
Proof. By compactness, there exists a finite subcover $\{U_1, \dots, U_m\}$. Assume that $U_i \neq X$ for some i (otherwise we are done) and set $C_i = X - U_i$ for $i \in \{1, \dots, m\}$. Assume that for every $n \geq 1$ there exists a point x_n such that $d(x_n, C_i) < 1/n$ for all i. Since X is compact and since compact metric spaces are sequentially compact, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$, with limit x. This x is a limit point of the closed set $\bigcap_i C_i$, and therefore $x \in \bigcap_i C_i$. On the other hand, $\bigcap_i C_i = X - \bigcup_i U_i = \emptyset$, a contradiction. It follows that there exists $\varepsilon > 0$ such that for all $x \in X$, there is a C_i with $\operatorname{dist}(x, C_i) \geq \varepsilon$. In other words, every open ball B_ε of radius ε is contained in some U_i, and hence also every set with diameter $< \varepsilon$. ☐

See Wikipedia, or *Topology: A first course* by J. Munkres, for a different proof. In our applications, we usually have a curve $\gamma : I \to X$ and a cover $X = \cup_\alpha U_\alpha$, and then deduce that we can subdivide the interval I by finding $t_0 < t_1 < \cdots < t_n$ such that each subinterval satisfies $\gamma([t_i, t_{i+1}]) \subset U_\alpha$ for some α. While the existence of such a subdivision follows from the above lemma, it can also be shown more directly.

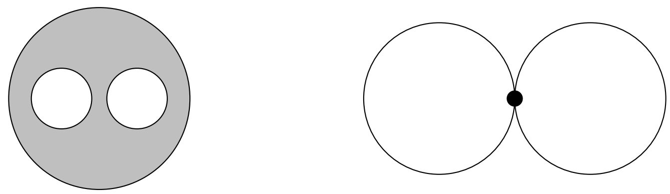
2 An explicit homotopy

In the lecture we used the fact that $\mathbb{R}^2$ without two points (say, $(-0.5, 0)$ and $(0.5, 0)$) is homotopy equivalent to the figure-eight $S^1 \vee S^1$:

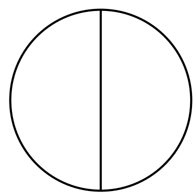




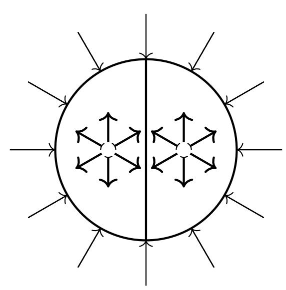
As mentioned early in the lecture, in topology one rarely describes such homotopy equivalences explicitly using formulas, once one is convinced that it is in principle possible. In this particular case, a hint to the fact that this is possible is given by the similar structure: the "outer space" in $\mathbb{R}^2$ can be retracted to a circle containing the two points, and the points can be expanded, so that the resulting spaces look very similar (with one of them having "thicker walls"):



As also mentioned in one of the early problem sheets, instead of finding an explicit deformation retract or homotopy from one space to the other, one can alternatively use a third space as a "proxy". In our case, it would seem convenient to use the following graph:



A retraction from $X = \mathbb{R}^2 - \{(-0.5, 0), (0.5, 0)\}$ to this graph (let's call it Θ) is easily described by mapping any point x outside the circle to $x/\|x\|$, any point on the left-side to the intersection of the line segment emanating from $(-0.5, 0)$ through x with Θ, and the same on the right half:



A deformation retract is then achived by attaching a straight-line homotopy to this retraction, i.e., if $r(x)$ is the retraction, then we define

$$F(x, t) = (1 - t)x + tr(x).$$

The graph Θ then deformation retracts to the figure-eight in the obvious way, by squashing the line in the middle towards a point (if still unsure, try to find equations for all the transformations above, in the same way that we found equations, for example, for describing the homotopy from D^n/S^{n-1} to S^n in an earlier sheet).

3 On the order of taking products and quotients

Suppose that $X = \tilde{X}/\sim$ is a quotient space, and we would like to construct a homotopy

$$F \colon X \times I \to Y.$$

A canonical approach is to first consider a homotopy $\tilde{F} \colon \tilde{X} \times I \to Y$ and then "factor" it through the quotient,

$$\begin{array}{ccc} \tilde{X} \times I & & \\ {\scriptstyle q\times\mathrm{Id}}\downarrow & \searrow^{\tilde{F}} & \\ X \times I & \xrightarrow{F} & Y \end{array}$$

We have seen cases of this before, for example when showing that the torus without a point deformation retracts to the figure-eight. We would consider the deformation of a square without a point, $\tilde{X} = I^2 - \{\mathrm{pt}\}$, to its boundary "frame" and then apply the torus identifications (pasting the left to the right boundary, and the upper to the lower) to get the figure-eight.

The problem with this type of construction is that it is not a priori obvious that the function F we arrive at is even continuous! The reason is that taking the product of a quotient is not the same as taking the quotient of a product, and hence we cannot be sure that $F^{-1}(U)$ is open, even if $\tilde{F}^{-1}(U)$ is open. In the situation above, however, this approach does work. To formalize this, we need the notion of an **identification map**: this is a surjective map $f \colon X \to Y$ such that $U \subset Y$ is open if and only if $f^{-1}(U)$ is open (Y has the *quotient topology*). For example, the quotient map $q \colon X \to X/\sim$ onto a quotient space is an identification map. Identification maps satisfy the following property (which you should try to prove).

Proposition 3.1. *A map $f \colon X \to Y$ is an identification map if and only if for every space Z and every function $g \colon Y \to Z$, $g \circ f$ is continuous if and only if g is continuous.*

$$\begin{array}{ccc} X & & \\ {\scriptstyle f}\downarrow & \searrow^{g\circ f} & \\ Y & \xrightarrow[g]{} & Z \end{array}$$

The following proposition solves all our problems (well, maybe not all).

Proposition 3.2. *If $f \colon X \to Y$ is an identification map, then $f \colon X \times I \to Y \times I$ is an identification map.*

Proof. We use Proposition **??**. Let $g \colon Y \times I \to Z$ be a function and consider the composition $h = g \circ (f \times \mathrm{Id}) \colon X \times I \to Z$. Clearly, if g is continuous, then so is h. Assume now that h is continuous. We would like to show that g is continuous.

Let $U \subset Z$ be an open set and let $(x,t) \in h^{-1}(U)$. Since h is continuous, $h^{-1}(U)$ is open in $X \times I$, and by the property of the product topology there exists an interval $[s,t] \subset I$ such that $h(\{x\} \times [s,t]) \subset U$. Define

$$V = \{y \in Y : g(\{y\} \times [s,t]) \subset U\}.$$

Since $f(x) \in V$, this set is not empty and the preimage under f is given by

$$f^{-1}(V) = \{x \in X \colon h(\{x\} \times [s,t]) \subset U\}.$$

Since f was assumed to be an identification map, V is open if and only if $f^{-1}(V)$ is open, so we will show that $f^{-1}(V)$ is open.

Note that if $x \in f^{-1}(V)$, then for all $s' \in [s,t]$,

$$(x, s') \notin C := (X \times [s,t]) \cap h^{-1}(Z - U).$$

Since C is closed, this means that for each $s' \in [s,t]$ there exists open sets $U_{s'}$ and $V_{s'}$ with $x \in U_{s'}$ such that

$$(U_{s'} \times V_{s'}) \cap C = \emptyset.$$

By compactness, there exist finitely many open sets U_i, V_i among those, such that $\bigcup_i V_i = [s,t]$. Then

$$\left(\bigcap_i U_i \times [s,t]\right) \cap C = \left(\bigcap_i U_i \times \bigcup_i V_i\right) \cap C = \emptyset.$$

It follows that $x \in \bigcap_i U_i \subset f^{-1}(V)$, and since $\bigcap_i U_i$ is open, that $f^{-1}(V)$ is open. $\square$

As an application of this result, we can show that a cone over a topological space is contractible (Exercise 7.3) or formally show that the torus without a point deformation retracts to a figure-eight (Exercise 6.1). The result holds for any locally compact space instead of I, see I.13.19 in *Geometry and Toplogy* by G. Bredon.

4 To be continued...

In the second part of this document, I will discuss Problem (6.5) in great detail, the computation of the fundamental group of projective space, and the solution of some assorted problems.