Lecture 14

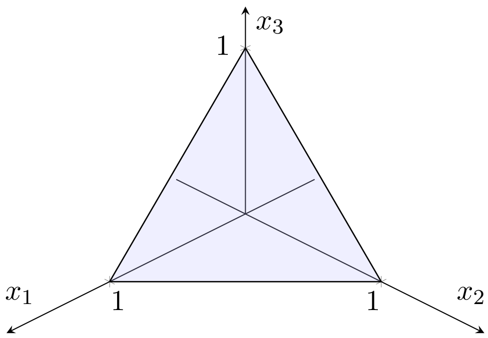
14.1 Applications of the Brouwer Fixed Point Theorem

An application of the Brouwer Fixed Point Theorem is to eigenvectors. Recall that an eigenvector of a matrix A is a vector v such that $Av = \lambda v$; the λ is called the eigenvalue. The following is a special case of the famous **Perron-Frobenius** Theorem, which has many applications, for example in graph theory or the theory of Markov chains.

Proposition 14.1. *Let $A \in \mathbb{R}^{3\times 3}$ be a matrix with only positive entries. Then A has an eigenvalue v consisting of only positive entries.*

Proof. Consider the **simplex**

$$\Delta = \{x \in \mathbb{R}^3 \colon x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 = 1\}.$$



Note that for any non-zero $x \in \mathbb{R}^3$ with non-negative entries, $x/\|x\|_1 \in \Delta$, where $\|x\|_1 = |x_1| + |x_2| + |x_3|$ is the 1-norm. Define the map

$$\varphi \colon \Delta \to \Delta, \quad x \mapsto \frac{Ax}{\|Ax\|_1}.$$

Since all the entries of A are positive, and all the entries of x are non-negative, all the coordinates of Ax are also positive, and we get a well-defined map from Δ to itself.

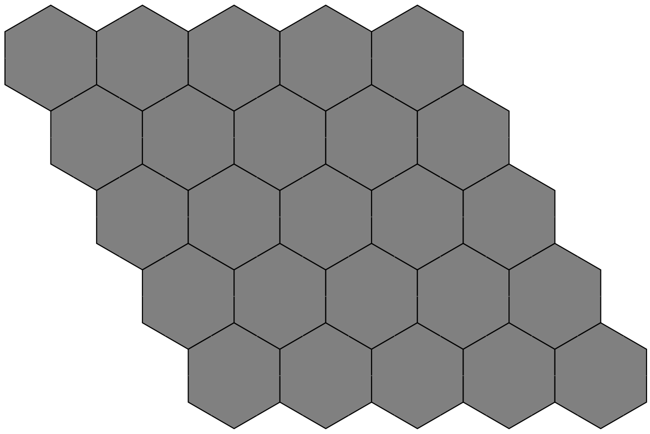
Since $\Delta \cong \mathbb{D}^2$ (exercise: show this!), this map has a fixed point: there exists $v \in \Delta$ such that $\varphi(v) = v$. For such a v, we have

$$Av = \|Av\|_1 v,$$

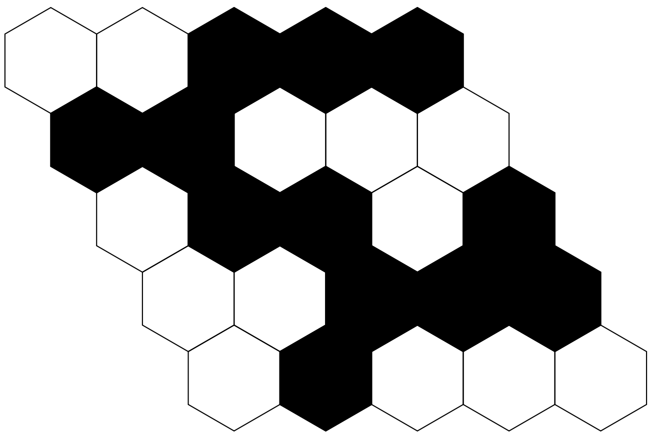
from which it follows that v is an eigenvalue. Since v is in the image of φ, its entries are all positive. $\square$

Disclaimer: The following treatment of the game HEX is strictly optional, this is not exam material.

Another application is to the game of HEX, invented by Piet Hein and John Nash independently in the 1940. The playing board consists of an $n \times n$ grid of hexagonal tiles.



Two players alternate in colouring hexagonal tiles black or white. The goal of the first player is to find a connected path from the bottom to the top consisting of black tiles, while the goal of the second player is to find a path from the left to the right consisting of white tiles. Assuming the players manage to fill the board, the result will look something like this:



It should be intuitively clear (and easy to show) that if there is a path of one colour connecting the bottom to the top end, then there cannot be a path of the opposite color connecting the left to the right end, and vice versa. What is less clear is that there is no possibility of having a draw: on of the players always wins. Put differently, it is impossible to colour the tiles black and white in such a way that no path exists connecting opposing sides.

Theorem 14.2. *Given an $n \times n$ board of hex with all the tiles coloured either black or white, there exists a path of one colour connecting the either the top and bottom sides, or the left and right sides.*

A proof of this Theorem using the Brouwer Fixed Point Theorem (and, in fact, a proof of the Fixed Point Theorem using this), can be found in the paper "Using Brouwer's fixed point theorem" by Björner, Matoušek and Ziegler.

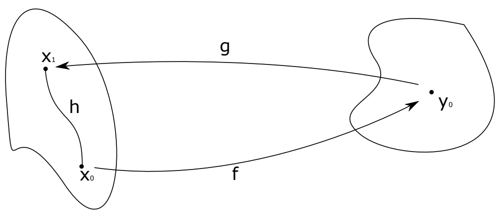
14.2 Homotopy invariance

So far we have seen that deformation retracts give rise to isomorphic fundamental groups. We next show that this holds more generally for homotopy equivalence.

Proposition 14.3. *Let $f\colon X \to Y$ be a homotopy equivalence. Then for any $x_0 \in X$, the induced map $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.*

Proof. Let $g\colon Y \to X$ be a homotopy inverse, so that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq Y$. Set $y_0 = f(x_0)$ and $x_1 = g(y_0)$. The composition $g \circ f$ thus gives rise to a homomorphism of fundamental groups

$$\begin{aligned}(g \circ f)_*\colon \pi_1(X, x_0) &\to \pi_1(X, x_1)\\ [\gamma] &\mapsto [g \circ f \circ \gamma].\end{aligned}$$



Let $K\colon X \times I \to X$ be a homotopy from Id_X to $g \circ f$, and define a path $h\colon I \to X$ from x_0 to x_1 by $h(t) = K(x_0, t)$. By Proposition 6.1 (Lecture 6), this path induces an isomorphism of fundamental groups

$$\begin{aligned}\beta_h\colon \pi_1(X, x_0) &\to \pi_1(X, x_1)\\ [\gamma] &\mapsto [\overline{h} * \gamma * h],\end{aligned}$$

where $\overline{h}(t) = h(1-t)$ is the inverse path, from x_1 to x_0. We claim that $\beta_h = (g \circ f)_*$. To show that these homomorphisms coincide, for any $[\gamma] \in \pi_1(X, x_0)$ we will construct a homotopy between $\overline{h} * \gamma * h$ and $g \circ f \circ \gamma$.

Define first a homotopy

$$h_t(s) = H(t,s) = \begin{cases} h(s) & \text{if } s \geq t \\ h(t) & \text{if } s \leq t \end{cases},$$

so that $h_1(s) = h(1) = x_1$ and $h_0(s) = h(s)$. In addition, define the homotopy

$$\gamma_t(s) = K(\gamma(s), t),$$

which consists in applying the homotopy from Id_X to $g \circ f$ to the loop γ. In particular, $\gamma_0 = \gamma$ and $\gamma_1 = g \circ f \circ \gamma$. Finally, consider the homotopy

$$\alpha_t(s) = \overline{h}_t * \gamma_t * h_t(s).$$

One checks directly from the definition that the endpoints of each of the concatenated paths coincide, and that $\alpha_0 = \overline{h} * \gamma * h$ and $\alpha_1 = g \circ f \circ \gamma$. We therefore have a homotopy $\overline{h} * \gamma * h \overset{\partial}{\simeq} g \circ f \circ \gamma$ and hence

$$\beta_h([\gamma]) = [\overline{h} * \gamma * h] = [g \circ f \circ \gamma] = (g \circ f)_*([\gamma]).$$

In particular, $(g \circ f)_* = g_* \circ f_*$ is an isomorphism, and therefore f_* is injective and g_* is surjective. Repeating the proof in the other direction (roles of f and g reversed), shows that f_* is surjetive and g_* is injective, thus finishing the proof that we have an isomorphism. $\square$