Lecture 15

The aim of this lecture is to prove the Borsuk-Ulam Theorem. In its two-dimensional version, the statement is as follows.

Theorem 15.1. *(Borsuk-Ulam) Let* $f\colon S^2 \to \mathbb{R}^2$ *be a map. Then there exists a point* $x \in S^2$ *with* $f(x) = f(-x)$.

The theorem thus states that for any continuous function from the sphere to $\mathbb{R}^2$ there are two antipodal points for which the function has the same value. One can interpret this as saying, for example, that there are always two antipodal points on the earth's surface with equal temperature and equal pressure (assuming these two are continuous functions). The theorem, which also holds in dimension $n \geq 2$, was first proven by Karol Borsuk, who in turn attributes the problem formulation to Stanislaw Ulam. It has remarkable ramifications and applications, an overview of which can be found in the book "Using the Borsuk-Ulam Theorem" by Jiři Matoušek.

15.1 The Borsuk-Ulam Theorem

To prove the Borsuk-Ulam Theorem we need a series of auxiliary results, which are interesting in their own right. These relate to the concepts of even and odd maps, and null homotopy.

An involution is a map $h\colon X \to X$ such that $h(h(x)) = x$. In this case, we often write $h(x) = -x$. Typical examples of spaces with involution are the spaces $\mathbb{D}^n$, S^n or $\mathbb{R}^n$, with $-x$ just the additive inverse of x.

Definition 15.2. Let X, Y be spaces with involution. A map $f\colon X \to Y$ is called **odd** if $f(-x) = -f(x)$, and **even** if $f(-x) = x$ for all $x \in X$.

Clearly, a map does not need to be either odd or even.

Example 15.3. The map $p_2\colon S^1 \to S^1$, $z \mapsto z^2$ is even. The sine function is odd, while the cosine function is even. The constant map $\mathrm{Id}_{\mathbb{R}^n}$ is odd.

Exercise 15.4. Show that the composition of odd maps is odd and that the composition of even maps with either even or odd maps is even.

Definition 15.5. A map $: X \to Y$ is called **null-homotopic** if f is homotopic to a constant map.

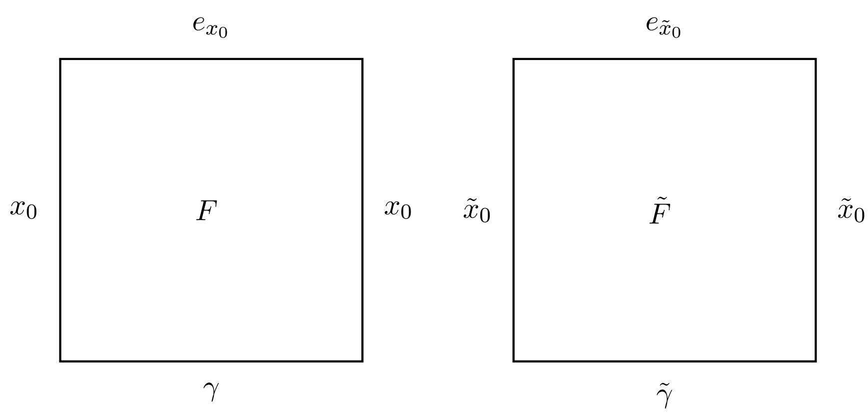
Definition 15.6. A pointed map $f\colon (X, x_0) \to (Y, y_0)$ is called null-homotopic relative to the basepoint if there is a homotopy $f\colon X \times I \to Y$ such that $f_0 = f$ and $f_1 = e_{y_0}$, with $f_t(x_0) = y_0$.

If f is null-homotopic relative to the basepoint, we write $f \overset{x_0}{\simeq} e$. If a map $f\colon X \to Y$ is null-homotopic to a constant map e_{y_0}, and if x_0 is such that $f(x_0) = y_0$, then this does not necessarily mean that the pointed map $f\colon (X, x_0) \to (Y, y_0)$ is null-homotopic. The added requirement is that each map f_t in the homotopy should map x_0 to y_0.

We will first show that odd maps from S^1 to S^1 cannot be null-homotopic. We will then use this to show that any odd map from S^2 to $\mathbb{R}^2$ has to have a root, and finally use this to establish Borsuk-Ulam Theorem, by noting that for a function $f\colon S^2 \to \mathbb{R}^2$, the function $f(x) - f(-x)$ is odd. Before we begin, we state a lemma that will be useful on several occasions.

Lemma 15.7. *Let $p\colon \tilde{X} \to X$ be a covering and let $\gamma\colon I \to X$ be a loop such that $\gamma \overset{\partial}{\simeq} e_{x_0}$. Let $\tilde{x}_0 \in p^{-1}(x_0)$ and $\tilde{\gamma}$ the lift of γ with $\tilde{\gamma}(0) = \tilde{x}_0$. Then $\tilde{\gamma} \overset{\partial}{\simeq} e_{\tilde{x}_0}$.*

Proof. Let $F\colon I \times I \to X$ be a homotopy with $f_0 = \gamma$ and $f_1 = e_{x_0}$. By the homotopy lifting property, there is a unique homotopy $\tilde{F}\colon I \times I \to \tilde{X}$ with $\tilde{f}_0 = \tilde{\gamma}$.



The paths $F(0, t) = x_0$ (left boundary), $F(1, t) = x_0$ (right boundary) and $F(s, 1) = x_0$ (upper boundary) are all constant paths. By Problem (4.7) (or the path lifting property), constant paths lift to constant paths, which implies that $\tilde{\gamma}$ is a loop at $\tilde{x}_0$ that is is homotopic to the constant loop $e_{\tilde{x}_0}$ via $\tilde{F}$. □

Proposition 15.8. *If $f\colon S^1 \to S^1$ is odd, then f is* not *null-homotopic.*

Proof. Assume that f is odd and that f is null-homotopic to a constant map, which without lack of generality we can assume to be e_1 (exercise: why?), via a homotopy

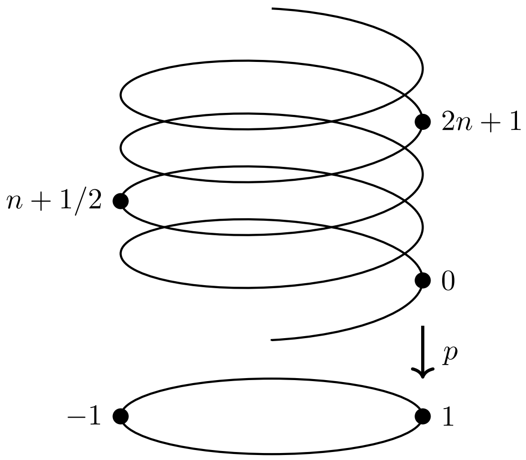
$F\colon S^1 \times I \to S^1$. Consider the cover $p\colon \mathbb{R} \to S^1$, $s \mapsto \exp(2\pi i s)$. We proceed in two steps.

(1) Use the oddity of f to construct a loop $\gamma\colon I \to S^1$ based at 1 in such a way that γ lifts to a path $\tilde{\gamma}$ from 0 to an odd endpoint $2n+1$.

Set $g = f/f(1)$. Clearly, this is again odd and has the property that $g(0) = g(1) = 1$. We can thus define a loop $\gamma\colon I \to S^1$ by setting $\gamma(s) = g(e^{2\pi i s})$. Since

$$e^{i\pi} = -1,$$

we get that $\gamma(s+1/2) = g(\exp(2\pi i s + \pi i)) = -\gamma(s)$ for $s \in [0, 1/2]$, where we used the fact that g is odd. In particular, $\gamma(1/2) = -1$. By applying path-lifting to γ, we get a curve $\tilde{\gamma}\colon I \to \mathbb{R}$ with $\tilde{\gamma}(0) = 0$ and $\tilde{\gamma}(1/2) = n + 1/2$ for some $n \in \mathbb{Z}$, since $\gamma(1/2) = -1$ and $p^{-1}(-1) = \{m + 1/2\colon m \in \mathbb{Z}\}$. We would like to show that, as we wind on up that road, we arrive at $\tilde{\gamma}(1) = 2n+1$.



Consider the two paths $\alpha, \beta\colon I \to \mathbb{R}$ given by $\alpha(s) = n + 1/2 + \tilde{\gamma}(s/2)$ and $\beta(s) = \tilde{\gamma}((s+1)/2)$. We then have

- $p \circ \alpha(s) = -\gamma(s/2) = \gamma((s+1)/2) = p \circ \beta$;
- $\alpha(0) = \beta(0) = n + 1/2$.

By the uniqueness of lifts, it follows that $\alpha = \beta$, and hence

$$2n + 1 = \alpha(1) = \beta(1) = \tilde{\gamma}(1).$$

This establishes the first part.

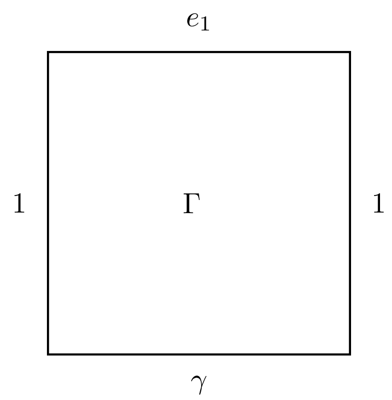
(2) Show that the lifted path $\tilde{\gamma}$ is a loop that is homotopic to the constant loop at 0.

Using the fact that F is null-homotopic, we construct a homotopy from γ to e_{x_0} as follows:

$$\Gamma\colon I \times I \to S^1$$
$$(s,t) \mapsto \frac{F(e^{2\pi i s}, t)}{F(1,t)},$$

where we use the division over the complex numbers. Clearly, at the boundaries

$$\Gamma(s,0) = \gamma(s), \quad \Gamma(s,1) = 1, \quad \Gamma(0,t) = \Gamma(1,t) = 1.$$



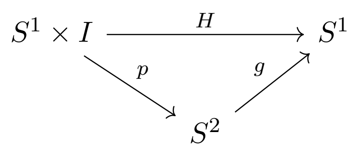
Thus Γ is a homotopy from γ to the constant loop at 1, and by Lemma 15.7, $\tilde{\gamma}$ is a null-homotopic loop. Therefore,

$$0 = \tilde{\gamma}(0) = \tilde{\gamma}(1) = 2n + 1,$$

which is not possible if $n \in \mathbb{Z}$. We get a contradiction to the assumption that f is null-homotopic, completing the proof. □

Corollary 15.9. *If $f\colon S^2 \to \mathbb{R}^2$ is odd, then there exists $x \in S^2$ such that $f(x) = 0$.*

Proof. Assume that f is odd, and that $f(x) \neq 0$ for all $x \in S^2$. The idea is to use f to define maps $g\colon S^1 \times I \to S^2$ and $p\colon S^2 \to S^1$, such that the composition $H = p \circ g\colon S^1 \times I \to S^1$ is a homotopy from and odd to a constant map, in contradiction to Proposition 15.8.



Define the map

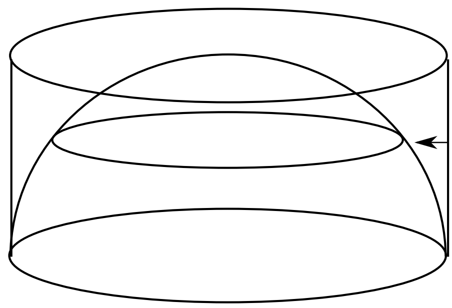
$$g\colon S^2 \to S^1,$$
$$x \mapsto \frac{f(x)}{\|f(x)\|}.$$

Since $g(-x) = f(-x)/\|f(-x)\| = -f(x)/\|f(x)\|$, g is again odd. Define the upper hemisphere

$$U = \{(x, y, z) \in S^2 \colon z \geq 0\},$$

and a map $p \colon S^1 \times I \to U$ by setting

$$p(e^{i\theta}, t) = (t\cos(\theta), t\sin(\theta), \sqrt{1 - t^2}).$$



It follows that p_0 is constant and p_1 embeds the circle S^1 into the equator $E = S^1 \times \{0\} \subset S^2$ (note that here, as usual, we identify $\mathbb{C}$ with $\mathbb{R}^2$ and $e^{i\theta}$ with $(\cos(\theta), \sin(\theta))$, and freely alternate between these representations). Finally, consider the homotopy

$$H \colon S^1 \times I \to S^1, \quad H(e^{i\theta}, t) = g \circ p.$$

Then $h_0 = g \circ p_0$ is a constant map and $h_1 = g \circ p_1$ is odd: this follows from the fact that both g and p_1 are odd. We therefore have a homotopy from an odd map to a constant map, in contradiction to Proposition 15.8. □

Proof of Theorem 15.1. Define the map

$$g(x) = f(x) - f(-x).$$

By definition, this is an odd map, so by Corollary 15.9 this has a zero, i.e., there exists an x such that $f(x) - f(-x) = 0$. □