Lecture 16

In this lecture we will compute the fundamental group of the torus (in any dimension) and of the n-dimensional sphere, for $n \geq 2$. As a consequence, we will see that

$$\mathbb{T}^n \not\simeq S^n.$$

16.1 Product spaces

The proof of the following is left as an exercise. The main ingredient is the observation that a map $f\colon Z \to X \times Y$ is continuous if and only if the compositions $p_X \circ f$ and $p_Y \circ f$ are continuous, where $p_X\colon X \times Y \to X$ and $p_Y\colon X \times Y \to Y$ are the projections onto X and Y, respectively.

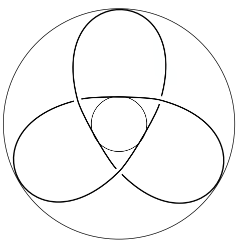
Proposition 16.1. *Let (X, x_0) and (Y, y_0) be pointed spaces. Then*

$$\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Example 16.2. Consider the torus $\mathbb{T}^2 = S^1 \times S^1$. Then

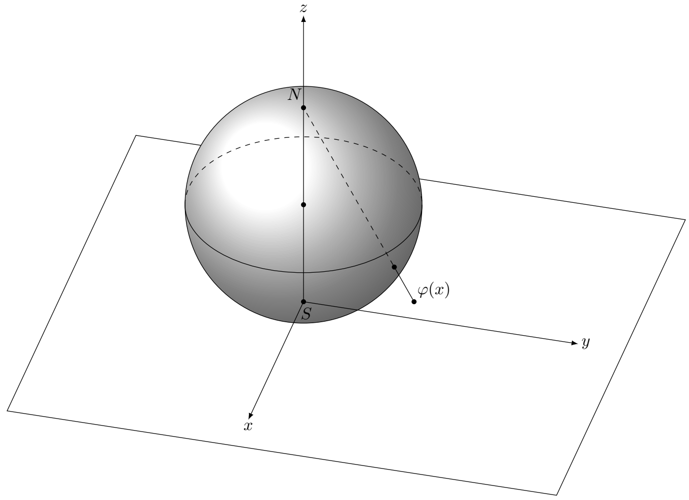
$$\pi_1(\mathbb{T}^2, (1,1)) = \mathbb{Z} \times \mathbb{Z}.$$

There are two types of loops in the embedded torus: that the fundamental group of the torus is $\mathbb{Z}^2$ should therefore not be surprising. Consider for example a loop that winds around one circle of the torus three times, and two times around the other. The resulting path is the trefoil know, one of many torus knots.



16.2 The fundamental group of the sphere S^n

The computation of the fundamental group of S^n for $n \geq 2$ uses the stereographic projection. Fix a point on S^n, for example the "north pole" $N = (0, \ldots, 0, 1)$.



The stereographic projection is a continuous map $\varphi\colon S^n - \{N\} \to \mathbb{R}^n$, by mapping a point x to the intersection of the line joining N and x with the hyperplane perpendicular to N touching the south pole $S = -N$.

Exercise 16.3. Derive the precise form of φ and its inverse φ^{-1}, and show that these are continuous maps.

Proposition 16.4. *For $n \geq 2$ and $x_0 \in S^n$, $\pi_1(S^n, x_0) \cong \{0\}$.*

Proof. As the stereographic projection gives a homeomorphism from the open set $U_1 = S^n - \{N\}$ to $\mathbb{R}^n$, and similarly from $U_2\colon S^n - \{S\}$ to $\mathbb{R}^n$, we see that the sphere can be written as a union of open sets

$$S^n = U_1 \cup U_2,$$

with $U_1 \cong \mathbb{R}^n$ and $U_2 \cong \mathbb{R}^n$. In addition, also using the stereographic projection, we see that $U_1 \cap U_2 \cong \mathbb{R}^n - \{0\}$, which is path-connected. Assume without lack of generality that $x_0 \in U_1 \cap U_2$ (as S^n is path-connected, the fundamental groups with different basepoints are all isomorphic).

Given a loop $\gamma\colon I \to S^n$, we have a cover $I = \gamma^{-1}(U_1) \cup \gamma^{-1}(U_2)$. By the Lebesgue covering lemma, we can find a subdivision $0 = t_0 < t_1 < \cdots < t_m = 1$ such that for every subinterval we have $\gamma([t_{i-1}, t_i]) \subset U_1$ or $\gamma([t_{i-1}, t_i]) \subset U_2$. Set $\gamma_i := \gamma|_{[t_{i-1}, t_i]}$ for $1 \leq i \leq m$. Then

$$\gamma = \gamma_1 * \cdots * \gamma_m.$$

If $\gamma([t_{i-1}, t_i]) \subset U_j$ and $\gamma([t_i, t_{i+1}]) \subset U_k$ for $j, k \in \{1, 2\}$ (the possible cases are that $j = k$ or $j \neq k$), then there exists a path α_i in $U_j \cap U_k$ connecting $\gamma(t_i)$ to x_0 (since that space is path-connected). Consider now the new path

$$\beta = (\gamma_1 * \alpha_1) * (\overline{\alpha}_1 * \gamma_2 * \alpha_2) * \cdots * (\overline{\alpha}_{m-1} * \gamma_m).$$

Since each of the $\alpha_{i-1} * \gamma_i * \overline{\alpha}_i$ is a loop, β is a concatenation of loops. Moreover, each of these loops is contained in one of U_1 or U_2 (or both), and since these spaces are homeomorphic to $\mathbb{R}^n$, each of these loops is homotopic to the constant loop e_{x_0}. Therefore,

$$\beta \overset{\partial}{\simeq} e_{x_0}.$$

But we also have that

$$\gamma \overset{\partial}{\simeq} \beta,$$

using a homotopy that moves each $\alpha_i * \overline{\alpha}_i$ to e_{x_0}. It follows that $[\gamma] = [e_0]$, and hence the fundamental group is the trivial group. □

Exercise 16.5. Find out where the argument breaks down for $n = 1$.

Corollary 16.6. *The n-torus $\mathbb{T}^n$ is not homotopic to the sphere S^n.*