Lecture 17

In this lecture we look at the relationship between isomorphism classes of covers and subgroups of the fundamental group. This is what is also known as **Galois correspondence**, due to its analogy to Galois theory, where one has field extensions instead of coverings and the Galois group instead of the fundamental group.

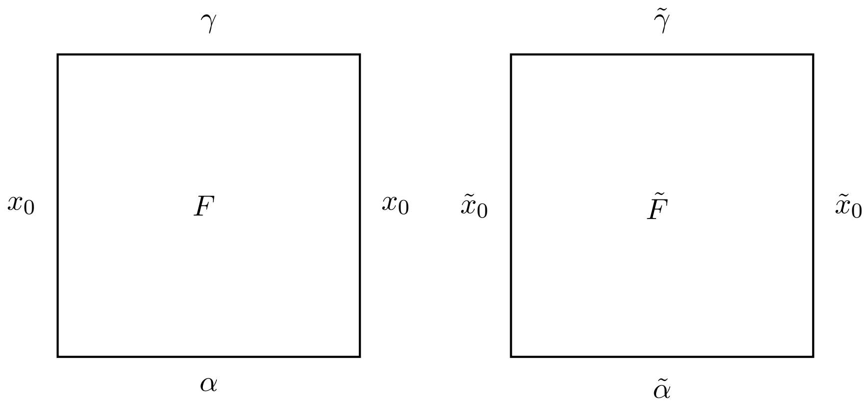
Proposition 17.1. *Let $p\colon \tilde{X} \to X$ be a covering, $x_0 \in X$, and $\tilde{x}_0 \in p^{-1}(x_0)$. Then:*

(a) The induced homomorphism $p_*\colon \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective;

(b) If $[\alpha] \in \pi_1(X, x_0)$ and $\tilde{\alpha}$ is the lift of α with $\tilde{\alpha} = \tilde{x}_0$, then $\tilde{\alpha}$ is a loop (i.e., $\tilde{\alpha}(1) = \tilde{x}_0$) if and only if $[\alpha] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Proof. **(a)** Assume that $[\tilde{\alpha}] \in \pi_1(\tilde{X}, \tilde{x}_0)$ maps to $[e_{x_0}]$, i.e., that $p \circ \tilde{\alpha} \overset{\partial}{\simeq} e_{x_0}$. Then by Lemma 15.7 (Lecture 15) we have that $\tilde{\alpha} \overset{\partial}{\simeq} e_{\tilde{x}_0}$.

(b) Clearly, if $\tilde{\alpha}$ is a loop, then $[\alpha] = p_*([\tilde{\alpha}])$, which shows the "only if" direction. For the "if" direction, assume that $[\alpha] = p_*([\tilde{\gamma}])$ for some $[\tilde{\gamma}] \in \pi_1(\tilde{X}, \tilde{x}_0)$. This means $\alpha = p \circ \tilde{\alpha} \overset{\partial}{\simeq} p \circ \tilde{\gamma} = \gamma$. Just as in the proof of Proposition 15.7, we get a homotopy from α to γ that fixes endpoints, and that lifts to a homotopy from $\tilde{\alpha}$ to $\tilde{\gamma}$.



As the left and right boundaries, $F(0, t)$ and $F(1, t)$, are constant (x_0), these lift to constant paths. Since the upper boundary $F(s, 1) = \tilde{\gamma}(s)$ is a loop at $\tilde{x}_0$, this means that the whole left and right boundaries are $\tilde{x}_0$, and therefore that $\tilde{\alpha}$ is a loop. □

The proposition shows that the image $p_*(\pi_1(\tilde{X}, \tilde{x}_0)$ is a subgroup of $\pi_1(X, x_0)$ that is isomorphic to $\pi_1(\tilde{X}, \tilde{x}_0)$. This may seem counterintuitive at first, as covering maps are (generally) surjective.

Example 17.2. Consider the d-fold covering $p_d \colon S^1 \to S^1$. Identifying the fundamental groups with $\mathbb{Z}$, the induced map is $n \mapsto d \cdot n$. Hence, $(p_d)_*(\pi_1(S^1, 1) \cong d\mathbb{Z}$.

Recall that if G is a group and $H \leq G$ a subgroup, the **index** of H in G, $[G \colon H]$, is the number of right-cosets $G/H = \{Hg\}_{g \in G}$.

Definition 17.3. Let $p \colon \tilde{X} \to X$ be a covering and assume that $\tilde{X}$ and X are path-connected. Then for any $x \in X$,

$$\deg(p) := |p^{-1}(x)|$$

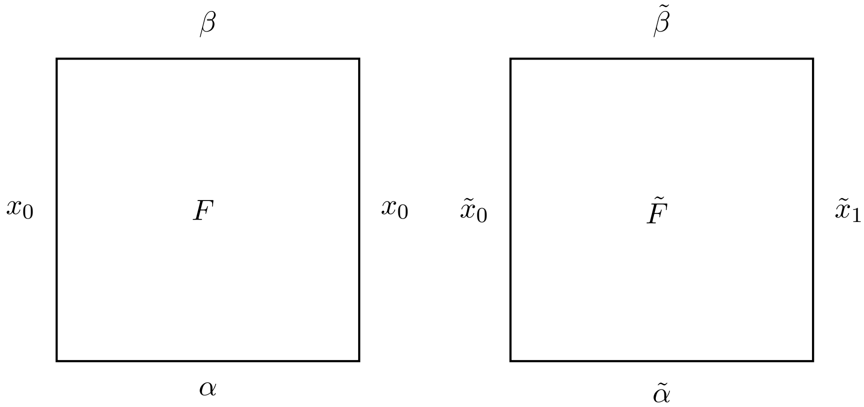
is called the **degree** of the covering.

Exercise 17.4. Show that this is well-defined. Verify whether the conditions of path-connectedness and the connectedness of $\tilde{X}$ can be relaxed.

Proposition 17.5. *Let $p \colon \tilde{X} \to X$ be a covering and assume that $\tilde{X}$ and X are path-connected. Let $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$. Then*

$$\deg(p) = [\pi_1(X, x_0) \colon p_*(\pi_1(\tilde{X}, \tilde{x}_0))].$$

Proof. Set $G = \pi_1(X, x_0)$ and $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Let $[\alpha] \in G$ and $\tilde{\alpha}$ a lift of α, starting at $\tilde{x}_0$ and ending at $\tilde{x}_1 \in p^{-1}(x_0)$. If β is another loop with $[\alpha] = [\beta]$, then by the same argument as in the proof of Proposition 17.1(b) (see the figure), β lifts to a path $\tilde{\beta}$ with the same endpoint $\tilde{x}_1$, so that the endpoint only depends on the class of the loop, and not the representative.



Let $[h] \in H$. Then by Proposition 17.1(b), h lifts to a loop $\tilde{h}$ in $\tilde{X}$, and $h * \alpha$ lifts to a path $\tilde{h} * \tilde{\alpha}$ from $\tilde{x}_0$ to $\tilde{x}_1$. From this we get a map from the set of right-cosets

$H[\alpha]$ to the endpoints of lifts $\tilde{\alpha}$:

$$\Phi\colon G/H \to p^{-1}(x_0)$$
$$H[\alpha] \mapsto \tilde{\alpha}(1).$$

We need to show that this map is injective and surjective.

For the injectivity, assume that $\Phi(H[\alpha]) = \Phi(H[\beta])$. Then $\tilde{\alpha}(1) = \tilde{\beta}(1)$, and hence $\tilde{\alpha} * \overline{\tilde{\beta}}$ is defined and is a loop at $\tilde{x}_0$ in $\tilde{X}$. It follows that

$$p_*([\tilde{\alpha} * \overline{\tilde{\beta}}]) = [p \circ \tilde{\alpha} * \overline{\tilde{\beta}}] = [p \circ \tilde{\alpha} * p \circ \overline{\tilde{\beta}}] = [\alpha] \bullet [\overline{\beta}] \in H,$$

from which we get that

$$H[\alpha] = H[\alpha][\overline{\beta}][\beta] = H[\beta].$$

For the surjectivity, we see that since $\tilde{X}$ is path connected, there exists a path from $\tilde{x}_0$ to any other point in $p^{-1}(x_0)$. Each such path projects to a loop α in X, and Φ maps the corresponding element $H[\alpha]$ in G/H to $\tilde{\alpha}(1)$. Therefore, Φ is a bijection. $\square$