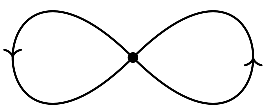
Lecture 18

In this lecture we will study free products of groups, a construction that is important in the study of the fundamental group of various spaces.

Consider for example the figure-eight

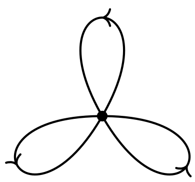


Definition 18.1. Let $\{(X_\alpha, x_\alpha)\}_\alpha$ be a collection of pointed topological spaces. The **wedge sum** of this collection is defined as

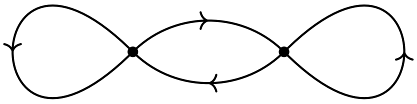
$$\bigvee_\alpha (X_\alpha, x_\alpha) = \bigsqcup_\alpha X_\alpha/(x_\alpha \sim x_\beta),$$

that is, the disjoint union of the X_α with the points x_α all identified.

Example 18.2. The figure-eight is $S^1 \vee S^1$ (we omit the basepoints from the notation when it is not important). The set bouquet is given by $S^1 \vee S^1 \vee S^1$.



$S^1 \vee S^1 \vee S^1$



$(S^1 \vee S^1) \vee S^1$

Note that $S^1 \vee S^1 \vee S^1$ is not the same as $(S^1 \vee S^1) \vee S^1$! The basepoints do play a role, in the latter we are identifying only two of them.

If we denote the two loops in the figure-eight $S^1 \vee S^1$ by a and b, and their inverses at $\overline{a}$ and $\overline{b}$, then intuitively, every loop can be written as a "word", for example

$$aaab\overline{b}b\overline{a}a\overline{a}bb$$

This means: go around a 3 times, then around b, around b backwards and again around b, etc. One can also see that a loop described like this can be "reduced" to a homotopic loop: loops of the form $b\overline{b}$ are homotopic to the constant loop, so we can replace them with e, and we can then remove the constant loops from the expression. It follows that the loop above is homotopic to a reduced loop of the form

$$a^3b\overline{a}b^2 \text{ or } a^3ba^{-1}b^2$$

From this observation one can conjecture the form of the fundamental group of the figure-eight: take two circles $A = S^1$ and $B = S^1$, joint them at a point x_0, take generators a and b of the fundamental groups of A and B (which are isomorphic to $\mathbb{Z}$), and describe the fundamental group of the figure-eight as the set of reduced words on a and b, where the inverse of a word is the word obtained by changing the order of letters and replacing every letter with its inverse (for example, the inverse of aba^2b^{-3} is $b^3a^{-2}b^{-1}a^{-1}$), and taking as multiplication the concatenation of words, followed by a reduction. We formalize this process next, using the concept of free product.

18.1 The free product of groups

Definition 18.3. Let $\{G_\alpha\}_\alpha$ be a collection of groups. A **word** on these groups is a finite sequence $g_1 \cdots g_m$ of elements of the $g_i \in G_{\alpha_i}$, and m is the length of the word. The empty word is denoted by ϵ. The **product** of two words is the concatenation,

$$(g_1 \cdots g_m) * (h_1 \cdots h_n) = g_1 \cdots g_m h_1 \cdots h_n.$$

Definition 18.4. A word $g = g_1 \cdots g_m$ is called **reduced** if $g_i \neq e_{\alpha_i}$ (the unit element of the group G_{α_i}) and for any two consecutive letters g_i, g_{i+1}, $\alpha_i \neq \alpha_{i+1}$ (that is, consecutive letters are not from the same group).

Given any word g on the groups $\{G_\alpha\}_\alpha$, we can **reduce** it to a reduced word g' as follows.

(a) If $g_i = e_{\alpha_i}$, then remove it from g;

(b) If $\alpha_i = \alpha_{i+1}$, then replace $g_i g_{i+1}$ with the group element $g_i \cdot g_{i+1}$ from G_{α_i}.

As every such operation reduces the length of the word by one, the process has to terminate. Moreover, a word is reduced if and only it can't be reduced further by the above two operations.

Remark 18.5. A word g can be reduced to a word g' in different ways, depending on the order in which the operations are applied. It is not yet obvious that every word reduces to a *unique* reduced word.

On the set of reduced words we can define a multiplication as follows. Given reduced words $g = g_1 \dots g_m$ and $h = h_1 \cdots h_n$, construct a new reduced word $g \bullet h$ by taking the concatenation $g * h$, and then reducing the word as follows:

- If g_m and h_1 are in different groups, then define $g \bullet h = g * h$, which is already reduced.
- If g_m and h_1 are in the same group, replace $g_m h_1$ by the product $g_m \cdot h_1$.
- If $g_m \cdot h_1 = e$ (the unit element of the corresponding group), then remove this from the word and repeat the procedure with g_{m-1} and h_2.

This process eventually leads to a reduced word, denoted by $g \bullet h$. Define the set

$$*_\alpha G_\alpha = \{ \text{ reduced words on } \{G_\alpha\}_\alpha \}$$

Theorem 18.6. *The pair $(*_\alpha G_\alpha, \bullet)$ is a group, called the **free product** of $\{G_\alpha\}_\alpha$. The unit element is the empy word $\epsilon = []$, and the inverse of an element $g_1 \cdots g_m$ is $g_m^{-1} \cdots g_1^{-1}$.*

Checking that the inverse has the given form is straight-forward. Checking associativity of the operation requires some work.