Lecture 20

Note that every group G_α is a subgroup of $*_\alpha G_\alpha$, via the inclusion that maps g to the word consisting only of g for $g \neq e$, and e to the empty word. Let

$$\iota_\alpha \colon G_\alpha \hookrightarrow *_\alpha G_\alpha$$

denote this inclusion. The free product of a collection of groups $\{G_\alpha\}_\alpha$ satisfies the following universal property.

Lemma 20.1. *Let $\{\varphi_\alpha\}_\alpha$ be a collection of group homomorphisms $\varphi_\alpha \colon G_\alpha \to G$. Then there exists a unique map*

$$*_\alpha\varphi_\alpha \colon *_\alpha G_\alpha \to G$$

such that $(*_\alpha\varphi_\alpha) \circ \iota_\alpha = \varphi_\alpha$.

Proof. Define

$$(*_\alpha\varphi_\alpha)(g_1 \cdots g_m) = \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_m}(g_m), \tag{20.1}$$

where we assumed that $g_i \in G_{\alpha_i}$. This clearly satisfies the property $*_\alpha\varphi_\alpha \circ \iota_\alpha = \varphi_\alpha$. Moreover, since every φ_α is a group homomorphism, for $g_i, g_{i+1} \in G_\alpha$ we get $\varphi(g_i)\varphi(g_{i+1}) = \varphi(g_i g_{i+1})$ and $\varphi(e_\alpha) = e$, so that $*_\alpha\varphi_\alpha$ is compatible with the operations bringing a word into reduced form. Therefore, $*_\alpha\varphi_\alpha(g \bullet h) = \varphi(g)\varphi(h)$ and we have a group homomorphism. The requirement that the restriction to the G_α satisfies $*_\alpha\varphi_\alpha \circ \iota_\alpha = \varphi_\alpha$ leaves one with no other choice than to define the homomorphism as in 20.1. ☐

20.1 The Seifert-van Kampen Theorem

We now apply the free product to topology. The goal is to reduce the computation of the fundamental group of an open cover to the fundamental groups of the individual sets in the cover.

Let $X = \bigcup_\alpha A_\alpha$ be an open cover and denote by $\iota_\alpha \colon A_\alpha \hookrightarrow X$ and the inclusion maps. Assume that $x_0 \in \bigcap_\alpha A_\alpha$. The inclusion maps induce maps

$$(\iota_\alpha)_* \colon \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$$

of the fundamental groups with base x_0. By Lemma 20.1, these maps induce a map

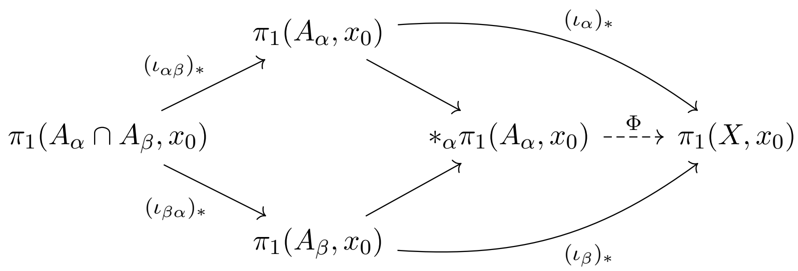
$$\Phi = *_\alpha(\iota_\alpha)_*\colon\ *_\alpha\, \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0),$$

and these maps are compatible with the inclusion $i_\alpha\colon \pi_1(A_\alpha, x_0) \hookrightarrow *_\alpha\pi_1(A_\alpha, x_0)$, in that $*_\alpha(\iota_\alpha)_* \circ i_\alpha = (\iota_\alpha)_*$.

It is relatively easy to show that if the pairwise intersections $A_\alpha \cap A_\beta$ are path-connected, then the induced map Φ is surjective. In general, however, it will not be injective: the reason is that loops in $A_\alpha \cap A_\beta$ are accounted for twice in $*_\alpha\pi_1(A_\alpha, x_0)$, once as an element of $\pi_1(A_\alpha, x_0)$ and once as an element of $\pi_1(A_\beta, x_0)$. To remedy this, we have to factor such loops out, and for this we need to study the inclusion

$$\iota_{\alpha\beta}\colon A_\alpha \cap A_\beta \to A_\alpha,$$

with the induced maps $(\iota_{\alpha\beta})_*$ of fundamental groups. The whole setup is summarised in the following "Seifert-van Kampen" commutative diagram:



Note that every $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$ is represented in $*_\alpha\pi_1(A_\alpha, x_0)$ as $(\iota_{\alpha\beta})_*(\omega)$, and as $(\iota_{\beta\alpha})_*(\omega)$. Define the set

$$U = \{(\iota_{\alpha\beta})_*(\omega)(\iota_{\beta\alpha})_*(\omega)^{-1}\colon \alpha, \beta, \omega \in \pi_1(A_\alpha \cap A_\beta, x_0)\},$$

and let N be the *normal closure* of U, i.e., the smallest normal subgroup of $*_\alpha\pi_1(A_\alpha, x_0)$ containing U. Recall that a subgroup $H \subset G$ is called *normal* if it is closed under conjugation: $gHg^{-1} = H$ for $g \in G$. We can now formulate the Seifert-van Kampen Theorem.

Theorem 20.2. *(Seifert-van Kampen) Let $X = \bigcup_\alpha A_\alpha$ be a cover with open sets and assume $x_0 \in \bigcap A_\alpha$. Then:*

(I) *If for all α, β, $A_\alpha \cap A_\beta$ is path-connected, then the map*

$$\Phi = *_\alpha(\iota_\alpha)_*\colon\ *_\alpha\, \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$$

is surjective.

(II) *If in addition for every α, β, γ the intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then* $\ker \Phi = N$*, and hence*

$$\pi_1(X, x_0) \cong *_\alpha\pi_1(A_\alpha, x_0)/N.$$

Before proving this theorem, we discuss a bit what it means. Any collection of loops $[\gamma_i] \in \pi_1(A_{\alpha_i}, x_0)$ gives rise to a loop $(\iota_{\alpha_i})_*([\gamma_i]) = [\iota_{\alpha_i}(\gamma_i)]$ in $\pi_1(X, x_0)$, and omitting the inclusion map we can simply denote it by $[\gamma_i] \in \pi_1(X, x_0)$. The induced map from the free product then looks as follows:

$$\begin{aligned}\Phi\colon\ *_\alpha\, \pi_1(A_\alpha, x_0) &\to \pi_1(X, x_0)\\ [\gamma_1]\cdots[\gamma_m] &\mapsto (\iota_{\alpha_1})_*([\gamma_1])\cdots(\iota_{\alpha_m})_*([\gamma_m]) = [\gamma_1 * \cdots * \gamma_m],\end{aligned}$$

where in the last line we consider γ_i as a loop in X. Thus for the first part, the surjectivity, we need to derive that every loop in X based at x_0 "factors" as a concatenation of loops $\gamma_1, \ldots, \gamma_m$, with each of these in one A_α. This is reminiscent of the derivation of the fundamental group of S^n for $n \geq 2$.

The fact that this map is *not injective* has to do with the fact that such a factorization is not unique: if γ is a loop in $A_\alpha \cap A_\beta$, then it is represented in $\pi_1(A_\alpha, x_0)$ as $(\iota_{\alpha\beta})([\gamma])$, and in $\pi_1(A_\beta, x_0)$ as $(\iota_{\beta\alpha}([\gamma])$. Each of these can appear as a letter in a word in $*_\alpha\pi_1(A_\alpha, x_0)$, and replacing one with the other in this word will not change the image of the word under Φ: the quotient $(\iota_{\alpha\beta})([\gamma])(\iota_{\beta\alpha})([\gamma])^{-1}$ is therefore in the *kernel* of Φ. The second part of the Seifert-van Kampen Theorem thus tells us that "factoring out" this kernel gives an isomorphism.

Example 20.3. Consider the sphere S^n and the cover U_1, U_2 consisting of the open sets by removing the north and the south pole, respectively. The intersection $U_1 \cap U_2$ is path-connected, so we have a surjective map

$$\Phi\colon \pi_1(U_1, x_0) * \pi_1(U_2, x_0) \to \pi_1(S^{n-1}, x_0).$$

Since U_1 and U_2 are contractible, the fundamental groups are the trivial group, and it follows that $\pi_1(S^n, x_0)$ is also the group with one element. The argument does not extend to S^1, since $U_1 \cap U_2$ is disconnected.