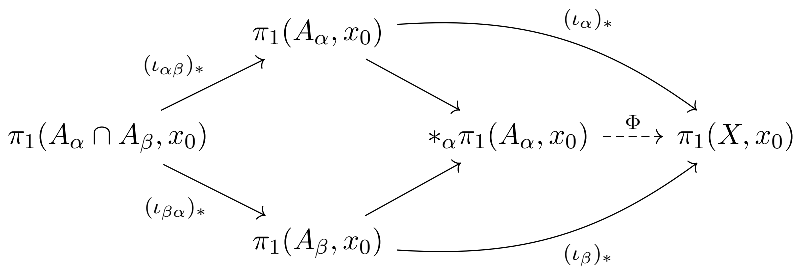
Lecture 21

Recall the setting of the Seifert-van Kampen theorem. Let $X = \bigcup_\alpha A_\alpha$ be an open cover and denote by $\iota_\alpha\colon A_\alpha \hookrightarrow X$ and $\iota_{\alpha\beta}\colon A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ the inclusion maps. Assume that $x_0 \in \bigcap_\alpha A_\alpha$. Then the inclusion maps induce maps between fundamental groups, as illustrated in the following commutative diagram:



Explicitly, each element of $*_\alpha \pi_1(A_\alpha, x_0)$ is a reduced word $[\gamma_1]\cdots[\gamma_m]$, with $[\gamma_i] \in \pi_1(A_{\alpha_i}, x_0)$, no γ_i the trivial loop, and $\alpha_i \neq \alpha_{i+1}$ for $1 \leq i < m$. The induced map Φ is defined by

$$\Phi([\gamma_1]\cdots[\gamma_m]) = (\iota_{\alpha_1})_*([\gamma_1]) \bullet \cdots \bullet (\iota_{\alpha_m})_*([\gamma_m]) = [\gamma_1 * \cdots * \gamma_m],$$

where in the last line we consider γ_i as a loop in X (formally, $\iota_{\alpha_i} \circ \gamma_i$). Recall that the subgroup $N \leq *_\alpha \pi_1(A_\alpha, x_0)$ was defined as the normal subgroup generated by elements of the form $(\iota_{\alpha\beta})_*(\gamma)(\iota_{\beta\alpha})_*(\gamma)^{-1}$. It is important to note that the expression $(\iota_{\alpha\beta})_*(\gamma)(\iota_{\beta\alpha})_*(\gamma)^{-1}$ cannot be simplified in $*_\alpha \pi_1(A_\alpha, x_0)$ unless $\omega = e$: the formal reason is that $\pi_1(A_\alpha, x_0)$ and $\pi_1(A_\beta, x_0)$ are simply different groups, and considered as subgroups of $*_\alpha \pi_1(A_\alpha, x_0)$ they have only the empty word as common element.

Theorem 21.1. *(Seifert-van Kampen) Let $X = \bigcup_\alpha A_\alpha$ be a cover with open sets and assume $x_0 \in \bigcap A_\alpha$. Then:*

(I) *If for all α, β, $A_\alpha \cap A_\beta$ is path-connected, then the map*

$$\Phi = *_\alpha (\iota_\alpha)_*\colon\ *_\alpha\, \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$$

is surjective.

(II) *If in addition for every α, β, γ the intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then* $\ker \Phi = N$*, and hence*

$$\pi_1(X, x_0) \cong *_\alpha \pi_1(A_\alpha, x_0)/N.$$

The first part is a consequence of the following lemma, which was already used in the derivation of the fundamental group of the sphere S^n for $n \geq 2$.

Lemma 21.2. *Let $X = \bigcup_\alpha A_\alpha$ be an open cover of a topological space, assume that $A_\alpha \cap A_\beta$ is path connected for all α, β and that $x_0 \in \bigcap A_\alpha$. Then every loop γ in X factors as*

$$[\gamma] = [\gamma_1] \bullet \cdots \bullet [\gamma_m],$$

with γ_i a loop in A_{α_i}.

Proof. Let $\gamma \colon I \to X$ be given, and consider the open cover $I = \bigcup_\alpha \gamma^{-1}(A_\alpha)$. Each of the $\gamma^{-1}(A_\alpha)$ is the union of open intervals, giving a cover of I by open intervals. By compactness, there is a finite subcover of open intervals, each of which maps into one A_α. Taking the closure of these intervals and subdividing if necessary, we find a sequence

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

such that $\gamma([t_{i-1}, t_i]) \subset A_{\alpha_i}$ for some A_{α_i} and $1 \leq i \leq m$. In particular, for every end-point t_i we have that $\gamma(t_i) \in A_{\alpha_i} \cap A_{\alpha_{i+1}}$ for $1 \leq i < m$. It follows that for every $i \in \{1, \ldots, m-1\}$ there exists a path β_i from $\gamma(t_i)$ to x_0 in $A_{\alpha_i} \cap A_{\alpha_{i+1}}$, with inverse path $\overline{\beta}_i$. We can then consider the modified path

$$\tilde{\gamma} = \gamma_1 * \cdots * \gamma_m,$$

where the γ_i are loops based at x_0, defined as

$$\gamma_i = \begin{cases} \gamma|_{[t_0,t_1]} * \beta_1 & \text{if } i = 1 \\ \overline{\beta}_{i-1} * \gamma|_{[t_{i-1},t_i]} * \beta_i & \text{if } i \in \{2, \ldots, m-1\} \\ \overline{\beta}_{m-1} * \gamma_{[t_{m-1},t_m]} & \text{if } i = m \end{cases}$$

Since the combinations $\beta_i * \overline{\beta}_i$ are the trivial loop, we have $\gamma \overset{\partial}{\simeq} \tilde{\gamma}$, and hence $[\gamma] = [\tilde{\gamma}] = [\gamma_1] \bullet \cdots \bullet [\gamma_m]$. $\square$

Proof. (of Theorem 20.1 (I)) For part (I), let $[\gamma] \in \pi_1(X, x_0)$. By Lemma 20.2, we can write $[\gamma] = [\gamma_1] \bullet \cdots \bullet [\gamma_m]$, with each γ_i a loop in one specific A_{α_i}. Moreover, we can assume that every γ_i is not the trivial loop, and that $\alpha_i \neq \alpha_{i+1}$ for $1 \leq i < m-1$ (otherwise we can just join γ_i and γ_{i+1} to one loop). This means that when considering $[\gamma_i]$ as elements of $\pi_1(A_{\alpha_i}, x_0)$, the word

$$[\gamma_1] \cdots [\gamma_m]$$

is a reduced word in $*_\alpha \pi_1(A_\alpha, x_0)$. By definition (see ()),

$$\Phi([\gamma_1]\cdots[\gamma_m]) = (\iota_{\alpha_1})_*([\gamma_1]) \bullet \cdots \bullet (\iota_{\alpha_m})_*([\gamma_m]) = [\gamma_1 * \cdots * \gamma_m] = [\gamma],$$

which shows that the map is surjective provided the pairwise intersections are path-connected. □