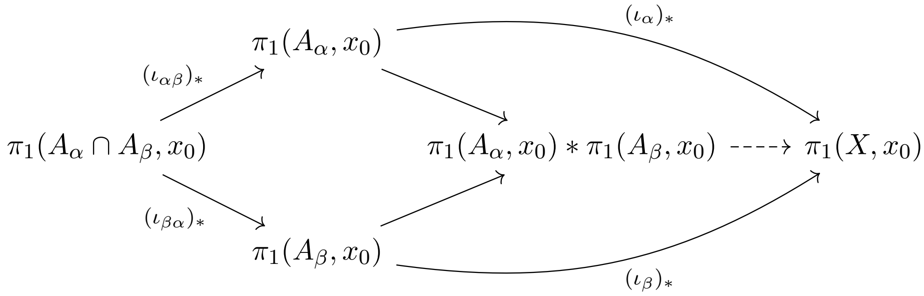
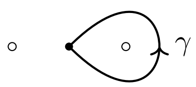
Lecture 22

Before outlining the proof of Part (II) of the Seifert-van Kampen Theorem, we give an example.

Example 22.1. Let $X = A_\alpha \cup A_\beta$ and consider the setting



Take $X = \mathbb{R}^2$, $A_\alpha = X - \{(1,0)\}$, $A_\beta = X - \{(-1,0)\}$ and $x_0 = (0,0) \in A_\alpha \cap A_\beta$. Let $\omega = [\gamma] \in \pi_1(A_\alpha \cap A_\beta, x_0)$ be a loop that winds around $(1,0)$.



The loop γ gives rise to different elements in each of the groups considered:

- $A_\alpha \cap A_\beta \simeq S^1 \vee S^1$ and the fundamental group $\pi_1(A_\alpha \cap A_\beta, x_0)$ is the free group on two generators a and b, with $[\gamma] = a$ one of them;
- $A_\alpha \simeq S^1$, and $(\iota_{\alpha\beta})_*([\gamma])$ is a generator of $\pi_1(A_\alpha, x_0) \cong \mathbb{Z}$;
- $A_\beta \simeq S^1$, but $(\iota_{\beta\alpha})_*([\gamma]) = e$, the constant loop in $\pi_1(A_\beta, x_0)$;
- $A_\alpha \cup A_\beta = \mathbb{R}^2$ and the image of γ under both $(\iota_\alpha)_* \circ (\iota_{\alpha\beta})_*$ and $(\iota_\beta)_* \circ (\iota_{\beta\alpha})_*$ is the unit element in the trivial group $\pi_1(\mathbb{R}^2, x_0)$.
- In $\pi_1(A_\alpha, x_0) * \pi_1(A_\beta, x_0)$, the concatenation of elements of $\pi_1(A_\alpha, x_0)$ with elements of $\pi_1(A_\beta, x_0)$ does not reduce, unless at least one of these is the unit element. In our case:

$$(\iota_{\alpha\beta})_*([\gamma]) \bullet (\iota_{\beta\alpha})_*([\gamma])^{-1} = (\iota_{\alpha\beta})_*([\gamma]) e_\beta = (\iota_{\alpha\beta})_*([\gamma]).$$

Note that even if the image of $[\gamma]$ in $\pi_1(A_\alpha, x_0)$ and in $\pi_1(A_\beta, x_0)$ "looks the same", we could still not cancel out concatenations of such elements in the free product $\pi_1(A_\alpha, x_0) * \pi_1(A_\beta, x_0)$, because as subgroups of this free product, these groups have only the empty word in common. In the free product, one can only concatenate elements from different groups, combining adjacent elements only if they come from the same group.

We outline a the proof of Part (II) of the Seifert-van Kampen Theorem. This follows from another lemma on the composition of loops. Assume again that $X = \bigcup_\alpha A_\alpha$ is an open cover with $x_0 \in \bigcap_\alpha A_\alpha$. Let $[f] \in \pi_1(X, x_0)$. A *factorization* of $[f]$ is a sequence

$$[f_1] \cdots [f_m]$$

such that $[f_i] \in \pi_1(A_{\alpha_i}, x_0)$ and $f \overset{\partial}{\simeq} f_1 * \cdots * f_m$. If $\alpha_i \neq \alpha_{i+1}$ for $1 \leq i < m$ and $[f_i] \neq e_{\alpha_i}$ for all i, then such a factorization simply gives a reduced word in $*_\alpha \pi_1(A_\alpha, x_0)$. We consider the following operations on such words:

- Reduction/expansion: If $[f_i], [f_{i+1}] \in \pi_1(A_\alpha, x_0)$, then

$$[f_1] \cdots [f_i][f_{i+1}] \cdots [f_m] \leftrightarrow [f_1] \cdots [f_i * f_{i+1}] \cdots [f_m]$$

- Exchange: If $[f_i] = (\iota_{\alpha\beta})_*(\omega)$ and $[g_i] = (\iota_{\beta\alpha})_*(\omega)$ for $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$, then

$$[f_1] \cdots [f_i] \cdots [f_m] \leftrightarrow [f_1] \cdots [g_i] \cdots [f_m]$$

We call two factorization *equivalent* if the can be related a sequence of reductions, expansions or exchanges. Note that in contrast to the reduction of a word, we allow to exchange elements of $\pi_1(A_\alpha, x_0)$ with element from $\pi_1(A_\beta, x_0)$ that arise from the same element in $\pi_1(A_\alpha \cap A_\beta, x_0)$.

Lemma 22.2. *Any two factorizations* $[f_1] \cdots [f_k]$ *and* $[f_1'] \cdots [f_\ell']$ *of* $[f] \in \pi_1(X, x_0)$ *are equivalent.*

Proof. Since $f \overset{\partial}{\simeq} f_1 * \cdots * f_k$ and $f \overset{\partial}{\simeq} f_1' * \cdots * f_\ell'$, there exists a homotopy $G\colon I \times I \to X$ with $g_0 = f_1 * \cdots * f_k$ and $g_1 = f_1' * \cdots * f_\ell'$. Using an approach similar to the proof of Part (I), we aim to decompose the homotopy by finding intermediate paths $\gamma_0, \ldots, \gamma_N\colon I \to X$ such that $\gamma_0 = g_0$, $\gamma_N = g_1$, and each γ_i has a factorization in such a way that the factorization of γ_{i+1} arises from that of γ_i by a reduction, expansion, or exchange operation.

(1) We first decompose $I \times I$ into rectangles. Consider the open cover $I \times I \subset \bigcup_\alpha G^{-1}(A_\alpha)$. By the product topology on the square, every open set $G^{-1}(A_\alpha)$ can be written as the union of open rectangles, and since $I \times I$ is compact, we have finitely many such rectangles. The closure of these rectangles covers $I \times I$, and after a common refinement, we can assume that we have a decomposition $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for each such rectangle

$[s_i, s_{i+1}] \times [t_j, t_{j+1}]$ there exists an α with $G([s_i, s_{i+1}] \times [t_j, t_{j+1}]) \subset A_\alpha$. Since $G^{-1}(A_\alpha)$ is open, there exists an $\epsilon > 0$ such that the small horizontal displacement $[s_i + \epsilon, s_{i+1} + \epsilon] \times [t_j, t_{j+1}]$ remains in $G^{-1}(A_\alpha)$. By shifting the rectangles in this way to the left or right, we can ensure that no point lies in more than three rectangles (see Figure 22.1).

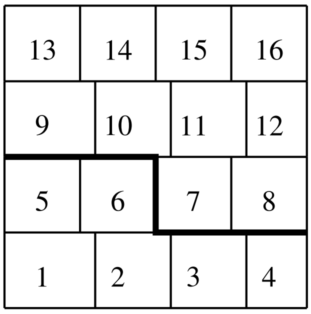
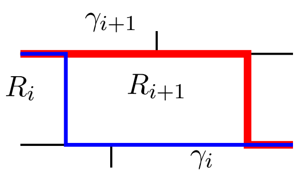


Figure 22.1: The subdivision of $I \times I$ and the path γ_6

(2) We next define paths along the rectangles. Let γ_i be the path from the left boundary to the right boundary that separates the rectangles $R_1, \dots, R_i$ from $R_{i+1}, \dots, R_N$. In particular, γ_0 is the lower boundary and γ_N the upper boundary, and γ_i and γ_{i+1} only differ on the boundary of R_{i+1} (γ_i goes under it, and γ_{i+1} above it). The homotopy from g_0 to g_1 gives a homotopy from γ_i to γ_{i+1}, by "pushing γ_i across R_{i+1}", i.e., applying G to a homotopy from the left-and-lower boundary of $\mathbb{R}_{i+1}$ to the right-and-upper boundary.

(3) We next associate a loop to every edge of a rectangle. Since every vertex v is in the intersection of at most three rectangles, it has the property that $G(v) \in A_\alpha \cap A_\beta \cap A_\gamma$ for some α, β, γ. By the assumption of path-connectedness, there exists a path h_v from $G(v)$ to x_0 in $A_\alpha \cap A_\beta \cap A_\gamma$. If R_i and R_{i+1} are adjacent rectangles with $G(R_i) \subset A_\alpha$ and $G(R_{i+1}) \subset A_\beta$, then their common boundary μ defines a path $G \circ \mu \in A_\alpha \cap A_\beta$ with endpoints v and w, and also a loop $\overline{h}(v) * (G \circ \mu) * h(w) \in A_\alpha \cap A_\beta$. If follows that every γ_i factors as a product of loops.

(4) We can now move from a factorization of γ_i to one of γ_{i+1} as follows. Replace any element in the factorization that arises from a loop corresponding to a boundary of R_{i+1} with an element in $\pi_1(A_\beta, x_0)$ (exchange operation). Then replace the loops corresponding to the left and lower boundary with loops corresponding to the upper



and right boundaries (reduction, replacement by homotopic loop, and expansion).

(5) The "boundary cases" may need to be treated separately. Altogether, we see that we can get from a factorization of g_0 to a factorization of g_1 by a sequence of exchanges, reductions, and expansions, thus showing that homotopic loops are equivalent in this sense. □

Proof. (Seifert-van Kampen, Part (II)) that map to an element $[f] \in \pi_1(X, x_0)$. Each of these represents a factorization of f, and by reductions and exchanges. Consider now the effect of an exchange on the map Φ.

Let $w = [f_1] \cdots [f_m]$ be a reduced word in $*_\alpha \pi_1(A_\alpha, x_0)$. If $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$ and $[f_i] = (\iota_{\alpha\beta})_*(\omega)$ and $[g_i] = (\iota_{\beta\alpha})_*(\omega)$, then replacing $[f_i]$ with $[g_i]$ (performing an exchange operation, and possibly reducing if necessary) gives a word v such that $w \bullet v^{-1} \in N$. The same is true if the word is expanded before the exchange and reduced after it. Therefore, if $w = [f_1] \cdots [f_k]$ and $v = [f'_1] \cdots [f'_\ell]$ are two elements in $*_\alpha \pi_1(A_\alpha, x_0)$ that arise from each other by exchanges, expansions and reductions, we have $wv^{-1} \in N$.

If $\Phi(w) = [e]$, then $e \overset{\partial}{\simeq} f_1 * \cdots * f_k$, so that w constitutes a factorization of e. Moreover, by Lemma 22.2 this factorization is equivalent to the empty factorization ϵ in the sense that they can be transformed into one another by a sequence of exchanges, expansions and reductions. It follows that $w \in N$, and hence $\ker \Phi \subset N$. The other inclusion is easy. □