Lecture 23

To be able to compute with, and compare, common topological spaces more effectively, we introduce the concept of a CW complex. CW complexes are topological spaces that can be assembled from simpler spaces by "glueing" cells together. Many, but not all, interesting topological spaces have the structure of a CW complex.

23.1 CW complexes

Definition 23.1. A **CW complex** is a topological space X that is built up inductively as follows.

1. The **zero-skeleton** X^0 is a discrete set;

2. Given X^{n-1}, a collection of closed *disks* $\{D_\alpha^n\}$ with $D_\alpha^n \cong B^n$, and $S_\alpha^{n-1} = \partial D_\alpha^n$, with **attaching maps**

$$\varphi_\alpha \colon S_\alpha^{n-1} \to X^{n-1},$$

define

$$X^n = (X^{n-1} \sqcup \bigsqcup_\alpha D_\alpha^n)/\sim,$$

where $\sim$ is the equivalence relation $x \sim \varphi_\alpha(x)$ for all $x \in S_\alpha^{n-1}$.

3. Define $X = \bigcup_n X^n$, equipped with the **weak topology**: a set $A \subset X$ is open if and only if $A \cap X^n$ is open *in* X^n for every n.

The disks D_α^n are called n-cells, and their interiors $e_\alpha^n = D_\alpha^n - S_\alpha^{n-1}$ are the open n-cells. The set X^n is called the n-skeleton of the CW complex. A CW complex is called **finite-dimensional** if $X = X^n$ for some n, and the largest n for which there are cells in the complex is called the **dimension** of the complex. A CW complex is called **finite** if it has only finitely many cells.

Example 23.2. A one-dimensional CW complex is called a (topological) **graph**. It consists of X^0 (the vertices), with X^1 arising by attaching the endpoints of intervals D_α^1 to the vertices.

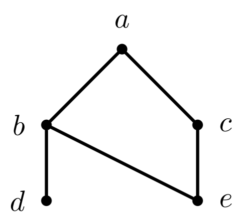


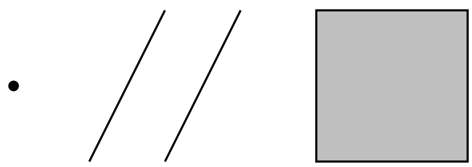
Figure 23.1: A graph.

A graph need not be finite. We can take, for example, as nodes $X^0 = \mathbb{Z}$, as edges copies of the unit interval I, and attaching maps φ_n defined by $\varphi_n(0) = n$ and $\varphi_n(1) = n + 1$. The resulting CW complex is homeomorphic to $\mathbb{R}$.

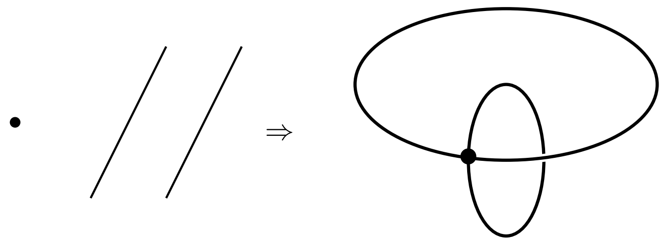
Exercise 23.3. Show that every connected graph is homeomorphic to a wedge of spheres $\vee_\alpha S^1$.

Example 23.4. We can fill in some of the closed areas of a graph, which gives rise to a two-dimensional CW complex. Other examples are polyhedra (the cube, the tetrahedron, etc.). The space $\mathbb{R}^n$ can be expressed as a CW complex in many different ways. The CW structure of a topological space is clearly not unique.

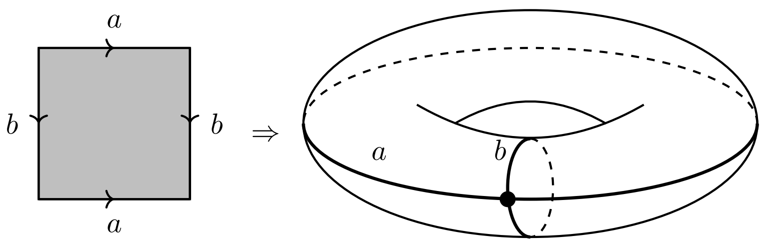
Example 23.5. The **torus** is an example of a two-dimensional CW complex. The ingredients are: one point $X^0 = \{x\}$, two line segments $\{I_1, I_2\}$, and one square D^2 (homeomorphic to the ball B^2).



For each of the line segments to the point by means of a map $\varphi_i \colon \partial I \to X^0$ (there is only one way of doing this).



We then attach the square to the resulting graph by a map $\varphi \colon \partial D^2 \to X^1$, mapping the upper and lower boundaries to one circle, and the left and right boundaries to the other circle. This is often visualized by drawing the square and labelling the edges in a way that indicates which edges are identified in which way:



Exercise 23.6. Show that the torus defined in this way is homeomorphic to $\mathbb{T}^2 = S^1 \times S^1$.

Exercise 23.7. Show that the sphere S^n is a CW complex with one 0-cell and one n-cell.