Recall the definition of a CW complex.

**Definition 24.1.** A **CW complex** is a topological space $X$ that is built up inductively as follows.

1. The **zero-skeleton** $X^0$ is a discrete set;

2. Given $X^{n-1}$, a collection of closed disks $\{D^a_n\}$ with $D^a_n \cong B^n$, and $S^a_{n-1} = \partial D^a_n$, with attaching maps $\varphi_a : S^a_{n-1} \to X^{n-1}$, define $X^n = (X^{n-1} \sqcup \bigsqcup_{\alpha} D^a_n) / \sim$, where $\sim$ is the equivalence relation $x \sim \varphi_a(x)$ for all $x \in S^a_{n-1}$.

3. Define $X = \bigcup_n X^n$, equipped with the **weak topology**: a set $A \subseteq X$ is open if and only if $A \cap X^n$ is open in $X^n$ for every $n$.

The disks $D^a_n$ are called closed $n$-cells, and their interiors $e^a_n = D^a_n - S^a_{n-1}$ are the open $n$-cells. The set $X^n$ is called the $n$-skeleton of the CW complex. A CW complex is called **finite-dimensional** if $X = X^n$ for some $n$, and the largest $n$ for which there are cells in the complex is called the **dimension** of the complex. A CW complex is called **finite** if it has only finitely many cells.

**Remark 24.2.** As a set, a CW complex is the union of the zero skeleton $X^0$ with disjoint open cells $e^a_n$.

**Definition 24.3.** A **subcomplex** of a CW complex $X$ is the closure in $X$ of a collection of open cells in $X$. 

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24.1 The Möbius strip and projective space

So far we have basic examples, such as graphs, the torus, and the sphere $S^n$. In this section we will revisit the projective plane $\mathbb{R}P^2$, and show that it can be characterized by gluing a disk to the boundary of a Möbius strip. We will then use this characterization as an alternative way of computing the fundamental group of $\mathbb{R}P^2$.

Example 24.4. The Möbius strip $M$ can be defined as $I \times I$ by identifying $(0, x)$ with $(1, 1-x)$ for $x \in I$.

![Figure 24.1: The Möbius strip](image)

There is one obvious CW complex structure on the Möbius strip: take 0 cells (the end points of $a$), three 1-cells (the line segment $a$ and the upper and lower boundaries of the rectangle), and one 2-cell, a rectangle itself. This is not the only way to describe the Möbius strip.

The Möbius strip has a circle at its centre, namely the image of $I \times \{1/2\}$ (since $(0, 1/2) \sim (1, 1/2)$). The Möbius strip deformation retracts to this circle by taking the homotopy on the rectangle, $\tilde{F} : (I \times I) \times I \to I \times I$, $((x, y), t) \mapsto (x, (1-t)(y-1/2)+1/2)$.

Since $1-[(1-t)(y-1/2)+1/2] = (1-t)(1-y-1/2)+1/2$, the homotopy carries over to a homotopy in the quotient. It follows that $\pi_1(M) \cong \mathbb{Z}$. The Möbius strip also has only one circle at its boundary, the image of $(I \times \{0\}) \times (I \times \{1\})$ under the quotient map.

Example 24.5. Real projective space $\mathbb{R}P^n$. Recall that

$$\mathbb{R}P^n = S^n/(x \sim -x),$$

the $n$-sphere with antipodal points identified (equivalently: the set of lines, that is, $\mathbb{R}^{n+1}$ with $x \sim y$ if $x = \lambda y$ for some $\lambda \in \mathbb{R}$). Let $q : S^n \to \mathbb{R}P^n$, $x \mapsto [x]$, be the quotient map. We can define a CW structure on $\mathbb{R}P^n$ recursively as follows. Consider the open set $U_0 = \{(x_0, x_1, \ldots, x_n) : x_0 \neq 0\}$. The set $\mathbb{R}P^n - U_0 = \{(0, x_1, \ldots, x_n)\}$ is homeomorphic to $\mathbb{R}P^{n-1}$. Moreover, since $q$ is a two-sheeted covering map, and the preimage $q^{-1}(U_0)$ consists of the disjoint union of the sets $\{x_0 > 0\}$ and $\{x_0 < 0\}$, each of which is the interior of an $n$-ball that maps
homeomorphically to $U_0$, and hence $U_0 \cong e^n$, an open disk. Setting $D^n = \{x_0 \geq 0\}$ and $S^{n-1} = \partial D^n = \{(0, x_1, \ldots, x_n)\}$, we get the two-fold covering

$$\varphi : S^{n-1} \to \mathbb{RP}^{n-1}$$

as attaching map (where we identified $\mathbb{RP}^{n-1} = \mathbb{RP}^n - U_0$), with $\mathbb{RP}^n$ arising as

$$\mathbb{RP}^{n-1} \sqcup D^n / (x \sim \varphi(x)).$$

We can continue this process recursively with $\mathbb{RP}^{n-1}$. As each step adds one open $n$-cell to the construction, we get a characterization of real projective space as

$$\mathbb{RP}^n = \{pt\} \cup e^1 \cup e^2 \cup \cdots \cup e^n,$$

with one open $n$-cell in each dimension.

In low dimensions, we have $\mathbb{RP}^0 = \{pt\}$, $\mathbb{RP}^1 = \mathbb{RP}^0 \sqcup D^1 / (0 \sim 1)$, which characterizes $\mathbb{RP}^1$ as a circle. For $\mathbb{RP}^2$, we attach a 2-cell by taking a disk $D^2$ and attaching the boundary circle $S^1$ to $\mathbb{RP}^1$ via the two fold covering $S^1 \to \mathbb{RP}^1$.

One way of thinking about $\mathbb{RP}^2$ is to take the closed upper hemisphere of a sphere $S^2$. Each point there corresponds to a a unique point in $\mathbb{RP}^2$, except at the boundary, where we have to identify antipodal points. But this makes the boundary an $\mathbb{RP}^1$. One can visualize the cell decomposition of $\mathbb{RP}^2$ as follows:

![Figure 24.2: Cell decomposition of $\mathbb{RP}^2$.](image-url)

The figure shows a 2-dimensional disk whose boundary disk is subdivided into cells that are identifies (the lines being identified along the arrow direction).

**Example 24.6.** ($\mathbb{RP}^2$ meets the Möbius strip). Consider the cell decomposition of $\mathbb{RP}^2$ as given in Figure 24.2, and let $X$ be the space obtained by removing a closed disk from the interior of $\mathbb{RP}^2$. 
Formally, we can describe $X$ as

$$X = S^1 \times I / (x, 1) \sim (-x, 1),$$

as $S^1 \times I$ describes the annulus, and the identification simply identifies antipodal points on one boundary of the annulus, but not on both. We claim that $X \cong M$, the Möbius strip. Visually, this can be seen by first “detaching” the annulus (keeping track of where the identifications happen), and then “reattaching” along $\gamma$, where we flip the upper rectangle around and rotate the lower rectangle by 180 degrees:
If we denote the concatenation $a = a_1 * a_2$, then we get exactly the characterization of Figure 24.1, with $\gamma$ the circle at the centre. As a consequence of this example, we see that we can obtain the projective plane by gluing a 2-cell $D^2$ to the boundary of a Möbius strip.

**Exercise 24.7.** Describe the homeomorphism $X \to M$ described above explicitly.

Given the above examples, we can compute the fundamental group of $\mathbb{R}P^2$ as follows. Recall the characterization of of $\mathbb{R}P^2$ from Figure 24.2, and denote by $e^2$ the interior of the disk. Consider a cover of $\mathbb{R}P^2$ as follows. Consider an open disk $B \subset e^2$ in $\mathbb{R}P^2$ and a closed disk $C \subset B$, and define $A = \mathbb{R}P^2 - C$ (see Figure 24.3). Then $\mathbb{R}P^2 = A \cup B$.

![Figure 24.3: An open cover of $\mathbb{R}P^2$](image)

Fix a base point $x_0 \in A \cap B$. Clearly, $\pi_1(B, x_0) = 1$, the trivial group, since $B$ is just an open disk. The intersection $A \cap B$ is homotopic to a circle, represented by a loop $\omega$, so that $\pi_1(A \cap B, x_0) = \langle [\omega] \rangle \cong \mathbb{Z}$. The set $A$, in turn, is the interior of a Möbius strip, as seen in Example 24.6, with $\gamma$ representing the inner circle. As seen in Example 24.4, $A$ deformation retracts to $\gamma$ (or, more precisely, to a circle homotopic to $\gamma$ but with basepoint $x_0$, see the figure), so that $\pi_1(A, x_0) \cong \langle [\gamma] \rangle \cong \mathbb{Z}$.

Since the fundamental group $\pi_1(B, x_0)$ is trivial, the free group $\pi_1(A, x_0) * \pi_1(B, x_0)$ is generated by $[\gamma]$. To get the fundamental group of $\mathbb{R}P^2$ using Seifert-van Kampen, we have to factor out elements that are multiples of

$$(\iota_{A \cap B})_*([\omega]),$$

where $\iota_{A \cap B}$ is the inclusion of $A \cap B$ in $A$. We can think of $\omega$ as the outer circle of a Möbius strip, and $\gamma$ as the inner circle. Going around $\omega$ once corresponds to going around $\gamma$ twice, so that

$$(\iota_{A \cap B})_*([\omega]) = [\gamma]^2.$$

By the Seifert-van Kampen Theorem,

$$\pi_1(\mathbb{R}P^2, x_0) \cong \langle [\gamma] \rangle / \langle [\gamma]^2 \rangle \cong \mathbb{Z} / 2\mathbb{Z}.$$