Lecture 24

Recall the definition of a CW complex.

Definition 24.1. A **CW complex** is a topological space X that is built up inductively as follows.

1. The **zero-skeleton** X^0 is a discrete set;

2. Given X^{n-1}, a collection of closed *disks* $\{D^n_\alpha\}$ with $D^n_\alpha \cong B^n$, and $S^{n-1}_\alpha = \partial D^n_\alpha$, with **attaching maps**

$$\varphi_\alpha \colon S^{n-1}_\alpha \to X^{n-1},$$

define

$$X^n = (X^{n-1} \sqcup \bigsqcup_\alpha D^n_\alpha)/\sim,$$

where $\sim$ is the equivalence relation $x \sim \varphi_\alpha(x)$ for all $x \in S^{n-1}_\alpha$.

3. Define $X = \bigcup_n X^n$, equipped with the **weak topology**: a set $A \subset X$ is open if and only if $A \cap X^n$ is open *in* X^n for every n.

The disks D^n_α are called closed n-cells, and their interiors $e^n_\alpha = D^n_\alpha - S^{n-1}_\alpha$ are the open n-cells. The set X^n is called the n-skeleton of the CW complex. A CW complex is called **finite-dimensional** if $X = X^n$ for some n, and the largest n for which there are cells in the complex is called the **dimension** of the complex. A CW complex is called **finite** if it has only finitely many cells.

Remark 24.2. As a set, a CW complex is the union of the zero skeleton X^0 with disjoint open cells e^n_α.

Definition 24.3. A **subcomplex** of a CW complex X is the closure in X of a collection of open cells in X.

24.1 The Möbius strip and projective space

So far we have basic examples, such as graphs, the torus, and the sphere S^n. In this section we will revisit the projective plane $\mathbb{RP}^2$, and show that it can be characterized by glueing a disk to the boundary of a Möbius strip. We will then use this characterization as an alternative way of computing the fundamental group of $\mathbb{RP}^2$.

Example 24.4. The Möbius strip M can be defined as $I \times I$ by identifying $(0, x)$ with $(1, 1-x)$ for $x \in I$.

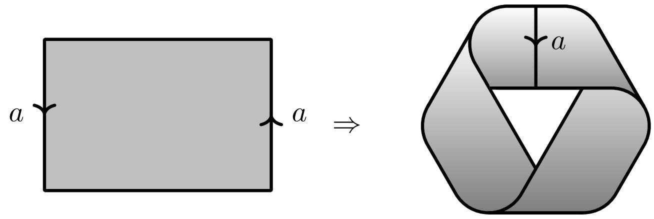


Figure 24.1: The Möbius strip

There is one obvious CW complex structure on the Möbius strip: take 0 cells (the end points of a), three 1-cells (the line segment a and the upper and lower boundaries of the rectangle), and one 2-cell, a rectangle itself. This is not the only way to describe the Möbius strip.

The Möbius strip has a circle at its centre, namely the image of $I \times \{1/2\}$ (since $(0, 1/2) \sim (1, 1/2)$). The Möbius strip deformation retracts to this circle by taking the homotopy on the rectangle,

$$\tilde{F} \colon (I \times I) \times I \to I \times I, \quad ((x, y), t) \mapsto (x, (1-t)(y - 1/2) + 1/2).$$

Since $1 - [(1-t)(y-1/2) + 1/2] = (1-t)(1-y-1/2) + 1/2$, the homotopy carries over to a homotopy in the quotient. It follows that $\pi_1(M) \cong \mathbb{Z}$. The Möbius strip also has *only one* circle at its boundary, the image of $(I \times \{0\}) \times (I \times \{1\})$ under the quotient map.

Example 24.5. Real projective space $\mathbb{RP}^n$. Recall that

$$\mathbb{RP}^n = S^n / (x \sim -x),$$

the n-sphere with antipodal points identified (equivalently: the set of lines, that is, $\mathbb{R}^{n+1}$ with $x \sim y$ if $x = \lambda y$ for some $\lambda \in \mathbb{R}$). Let $q \colon S^n \to \mathbb{RP}^n$, $x \mapsto [x]$, be the quotient map. We can define a CW structure on $\mathbb{RP}^n$ recursively as follows. Consider the open set $U_0 = \{[(x_0, x_1, \ldots, x_n)] \colon x_0 \neq 0\}$. The set $\mathbb{RP}^n - U_0 = \{[(0, x_1, \ldots, x_n)]\}$ is homeomorphic to $\mathbb{RP}^{n-1}$. Moreover, since q is a two-sheeted covering map, and the preimage $q^{-1}(U_0)$ consists of the disjoint union of the sets $\{x_0 > 0\}$ and $\{x_0 < 0\}$, each of which is the interior of an n-ball that maps homeomorphically to U_0, and hence $U_0 \cong e^n$, an open disk. Setting $D^n = \{x_0 \geq 0\}$ and $S^{n-1} = \partial D^n = \{(0, x_1, \ldots, x_n)\}$, we get the two-fold covering

$$\varphi\colon S^{n-1} \to \mathbb{RP}^{n-1}$$

as attaching map (where we identified $\mathbb{RP}^{n-1} = \mathbb{RP}^n - U_0$), with $\mathbb{RP}^n$ arising as

$$\mathbb{RP}^{n-1} \sqcup D^n/(x \sim \varphi(x)).$$

We can continue this process recursively with $\mathbb{RP}^{n-1}$. As each step adds one open n-cell to the construction, we get a characterization of real projective space as

$$\mathbb{RP}^n = \{pt\} \cup e^1 \cup e^2 \cup \cdots \cup e^n,$$

with one open n-cell in each dimension.

In low dimensions, we have $\mathbb{RP}^0 = \{\text{pt}\}$, $\mathbb{RP}^1 = \mathbb{RP}^0 \sqcup D^1/(0 \sim 1)$, which characterizes $\mathbb{RP}^1$ as a circle. For $\mathbb{RP}^2$, we attach a 2-cell by taking a disk D^2 and attaching the boundary circle S^1 to $\mathbb{RP}^1$ via the two fold covering $S^1 \to \mathbb{RP}^1$.

One way of thinking about $\mathbb{RP}^2$ is to take the closed upper hemisphere of a sphere S^2. Each point there corresponds to a a unique point in $\mathbb{RP}^2$, except at the boundary, where we have to identify antipodal points. But this makes the boundary an $\mathbb{RP}^1$. One can visualize the cell decomposition of $\mathbb{RP}^2$ as follows:

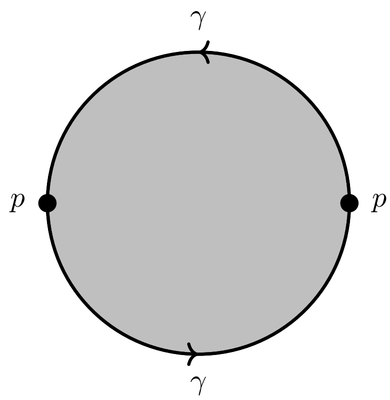
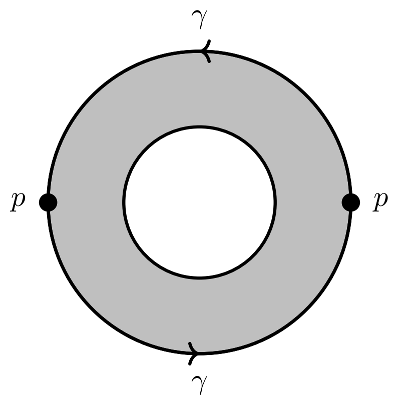


Figure 24.2: Cell decomposition of $\mathbb{RP}^2$.

The figure shows a 2-dimensional disk whose boundary disk is subdivided into cells that are identifies (the lines being identified along the arrow direction).

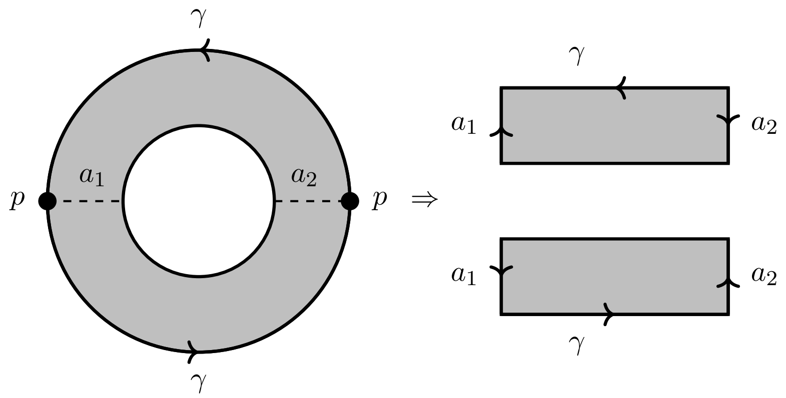
Example 24.6. ($\mathbb{RP}^2$ meets the Möbius strip). Consider the cell decomposition of $\mathbb{RP}^2$ as given in Figure 24.2, and let X be the space obtained by removing a closed disk from the interior of $\mathbb{RP}^2$.



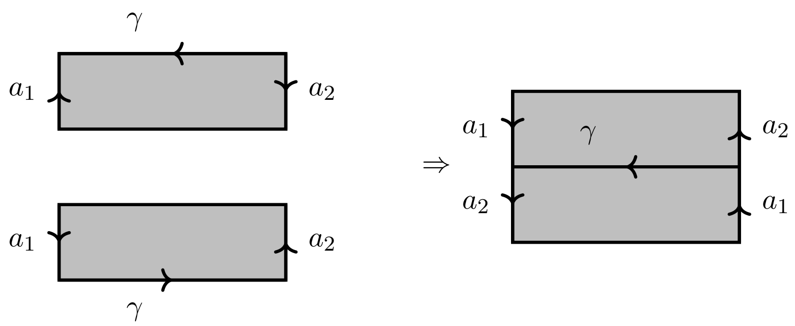
Formally, we can describe X as

$$X = S^1 \times I/(x,1) \sim (-x,1),$$

as $S^1 \times I$ describes the annulus, and the identification simply identifies antipodal points on one boundary of the annulus, but not on both. We claim that $X \cong M$, the Möbius strip. Visually, this can be seen by first "detaching" the annulus (keeping track of where the identifications happen),



and then "reattaching" along γ, where we flip the upper rectangle around and rotate the lower rectangle by 180 degrees:



If we denote the concatenation $a = a_1 * a_2$, then we get exactly the characterization of Figure 24.1, with γ the circle at the centre. As a consequence of this example, we see that we can obtain the projective plane by glueing a 2-cell D^2 to the boundary of a Möbius strip.

Exercise 24.7. Describe the homeomorphism $X \to M$ described above explicitly.

Given the above examples, we can compute the fundamental group of $\mathbb{RP}^2$ as follows. Recall the characterization of of $\mathbb{RP}^2$ from Figure 24.2, and denote by e^2 the interior of the disk. Consider a cover of $\mathbb{RP}^2$ as follows. Consider an open disk $B \subset e^2$ in $\mathbb{RP}^2$ and a closed disk $C \subset B$, and define $A = \mathbb{RP}^2 - C$ (see Figure 24.3). Then $\mathbb{RP}^2 = A \cup B$.

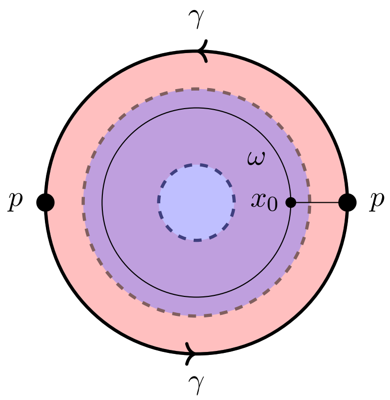


Figure 24.3: An open cover of $\mathbb{RP}^2$

Fix a base point $x_0 \in A \cap B$. Clearly, $\pi_1(B, x_0) = \mathbf{1}$, the trivial group, since B is just an open disk. The intersection $A \cap B$ is homotopic to a circle, represented by a loop ω, so that $\pi_1(A \cap B, x_0) = \langle [\omega] \rangle \cong \mathbb{Z}$. The set A, in turn, is the interior of a Möbius strip, as seen in Example 24.6, with γ representing the inner circle. As seen in Example 24.4, A deformation retracts to γ (or, more precisely, to a circle homotopic to γ but with basepoint x_0, see the figure), so that $\pi_1(A, x_0) \cong \langle [\gamma] \rangle \cong \mathbb{Z}$.

Since the fundamental group $\pi_1(B, x_0)$ is trivial, the free group $\pi_1(A, x_0) * \pi_1(B, x_0)$ is generated by $[\gamma]$. To get the fundamental group of $\mathbb{RP}^2$ using Seifert-van Kampen, we have to factor out elements that are multiples of

$$(\iota_{A\cap B})_*([\omega]),$$

where $\iota_{A\cap B}$ is the inclusion of $A \cap B$ in A. We can think of ω as the outer circle of a Möbius strip, and γ as the inner circle. Going around ω once corresponds to going around γ twice, so that

$$(\iota_{A\cap B})_*([\omega]) = [\gamma]^2.$$

By the Seifert-van Kampen Theorem,

$$\pi_1(\mathbb{RP}^2, x_0) \cong \langle [\gamma] \rangle / \langle [\gamma]^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}.$$