Lecture 25

25.1 Properties of CW complexes

Recall that we denoted closed cells of dimension n by D^n_α, their boundary by S^{n-1}_α, and $e^n_\alpha = D^n_\alpha - S^{n-1}_\alpha$ the open cell. To simplify notation, we will call the 0-cells, the points in X^0, e^0_α.

Given a CW complex X, for every closed cell D^n_α we have an inclusion map into the disjoint union of X^{n-1} with n-cells, that gives rise to a map into $X^n \subset X$ by applying the quotient map q to it,

$$\begin{tikzcd} & X^{n-1} \sqcup \bigsqcup_\alpha D^n_\alpha \arrow[d, "q"] & \\ D^n_\alpha \arrow[ur, hook, "\iota"] \arrow[r] & X^n \arrow[r, hook] & X \end{tikzcd}$$

where X^n arises by identifying $x \sim \varphi^n_\alpha(x)$ for points $x \in S^{n-1}_\alpha$. This gives rise to a **characteristic map**

$$\Phi^n_\alpha \colon D^n_\alpha \to X,$$

and the restriction of Φ^n_α to e^n_α is a homeomorphism to its image, also denoted by e^n_α. We can therefore characterize a CW complex as disjoint union of cells e^n_α.

Exercise 25.1. Show that the weak topology on X can be characterized by saying that $A \subset X$ is closed if and only if for each n, α, $(\Phi^n_\alpha)^{-1}(A)$ is closed in D^n_α.

Definition 25.2. A **subcomplex** of a CW complex X is a space A that is a union of cells e^α_n in X such that for every cell it also contains its closure.

We now study some important properties of CW complexes.

Proposition 25.3. *A compact topological subspace of a CW complex X is contained in a finite subcomplex.*

Proof. Let $C \subset X$ be a compact set, and assume that C intersects infinitely many cells e^n_α. Then there exists a sequence of points $S = \{x_1, x_2, \dots\} \subset C$ so that each x_i lies in a different cell. Using the characterization of the weak topology via the characteristic map one can show that S is closed in X. Moreover, as every subset of S is closed, the topology on S is the discrete topology. As a closed subset of C, S is compact, but any compact set in the discrete topology is finite, so S is finite. It follows that C is contained in finitely many cells. It remains to show that that a finite union of cells is contained in a finite subcomplex. This can be seen by induction on n. The statement is clearly true for $n = 0$, since a finite union of 0-cells is just a finite set of points. If $n \geq 1$, then for every e^n_α, the image of the attaching map $\varphi^n_\alpha \colon S^{n-1}_\alpha \to X^{n-1}$ is compact, hence contained in finite union of cells of dimension at most $n-1$, which by induction hypothesis are contained in a finite subcomplex A. Attaching D^n_α to this subcomplex gives a finite complex containing e^n_α. □

The letter 'C' in CW complex means **closure finiteness**: the closure of every open cell meets only finitely many other cells. The 'W' stands for weak topology.

Definition 25.4. A topological space is called **normal** if any two disjoint closed subsets have disjoint open neighbourhoods. A topological spaces is called a **Hausdorff space**, if any two distinct points have disjoint open neighbourhoods.

Proposition 25.5. *A CW complex is normal (and hence Hausdorff).*

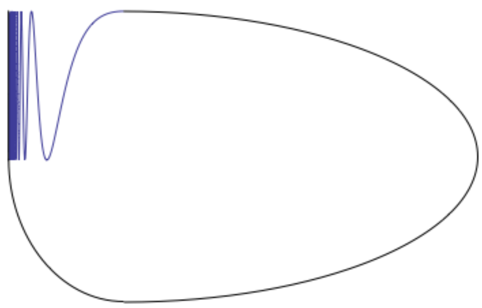
Definition 25.6. A topological space is called **locally contractible** if for every x and open neighbourhood U with $x \in U \subset X$ there exists an open set V with $x \in V \subset U$ such that V is contractible.

Example 25.7. Any open subset of $\mathbb{R}^n$ is contractible.

Example 25.8. Consider the Warsaw circle, defined as

$$W = \{(x, \sin(1/x) \colon x \in (0,1]\} \cup (\{0\} \times [-1,1]) \cup L,$$

where L is a curve jointing the first to sets in the union, with the subspace topology.



The Warsaw circle is not locally contractible. If it were, then it would be locally path connected (if V is a contractible neighbourhood of a point x, then any two points in V can be connected by a path via the homotopy between the identity on V and the retraction to a point in V). It is, however, not locally path connected. To see this, take any point on the piece $\{0\} \times I$, say $x = (0,0)$. Then every open neighbourhood of x of diameter less than 1 has infinitely many disconnected points. More precisely, if

$V = \{y \in W : \|y\| < \varepsilon\}$ for $\varepsilon < 1$, then the points $(1/n\pi, 0)$ for integers $n > 1/\varepsilon\pi$ are all in V, but are not connected by a path. Note however that W is path-connected!

Proposition 25.9. *CW complexes are* ***locally contractible****.*

Corollary 25.10. *The Warsaw circle is not a CW complex.*