Example Sheet 5

Please let me (Martin) know if any of the problems are unclear or have typos. Please turn in a solution to Exercise (5.2) or Exercise (5.3) by 14:00 on 8/11/2018, to the dropoff box in front of the undergraduate office. If you collaborate with other students, please include their names.

(5.1) Define the punctured plane to be $\mathbb{C}^\times = \mathbb{C} - \{0\}$. Show that the map $p\colon \mathbb{C}^\times \to \mathbb{C}^\times$ defined by $p(z) = z^2$ is a covering map. Explain why the squaring map on $\mathbb{C}$ itself is not a covering map.

(5.2) Show that for every homomorphism $\varphi\colon \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ there is a pointed map $f\colon (S^1, 1) \to (S^1, 1)$ so that $\varphi = f_*$. In other words, f induces φ.

(5.3) Suppose that $p\colon \tilde{X} \to X$ is a covering map, and suppose that $\tilde{X}$ is path-connected. Show, using the homotopy lifting property for paths, that $\tau \in \mathrm{Deck}(p)$ fixes a point of $\tilde{X}$ if and only if $\tau = \mathrm{Id}_{\tilde{X}}$.

(5.4) Suppose that $p\colon \mathbb{R} \to S^1$ is the usual covering map, namely $p(t) = \exp(2\pi i t)$. Give a complete proof that $\mathrm{Deck}(p) \cong \mathbb{Z}$.

(5.5) Show that there is no retraction $r\colon X \to A$ in any of the following cases. (Give short justifications of any fundamental group computations.)

- $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.
- $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.

(5.6) [Hard.] We say that a space X has the *fixed point property* if every map $f\colon X \to X$ has a fixed point. Define the *tripod* to be the set

$$T = \big\{ r \exp(2\pi i k/3) \in \mathbb{C} \colon r \in [0, 1], k \in \{0, 1, 2\} \big\}.$$

So T is a connected graph with three edges and four vertices. Prove the tripod has the fixed point property.