Example Sheet 6

Please let me (Martin) know if any of the problems are unclear or have typos. Please turn in a solution to one of Exercises (6.1), (6.2) or (6.3) by 14:00 on 15/11/2018, to the dropoff box in front of the undergraduate office. If you collaborate with other students, please include their names.

(6.1) Recall that $\mathbb{T}^2 = S^1 \times S^1$ is the torus. Fix any point $x_0 \in \mathbb{T}^2$ and show that $\mathbb{T}^2 - \{x_0\}$ deformation retracts to the figure-eight graph.

Hint: Use the description of the torus as a quotient of the square $I \times I$, by identifying the left with the right boundary, and the upper with the lower boundary (i.e., $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$). You can then argue with carefully drawn figures.

(6.2) Given topological spaces X and Y, equip $Z = X \times Y$ with the product topology. Let $p_X \colon X \times Y \to X$ and $p_Y \colon X \times Y \to Y$ be the projections of onto X and Y, respectively. Fix $x_0 \in X$ and $y_0 \in Y$, and let $(p_X)_* \colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0)$ and $(p_Y)_* \colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(Y, y_0)$ be the induced group homomorphisms.

Prove that the homomorphism

$$(p_X)_* \times (p_Y)_* \colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

is an isomorphism.

(6.3) The *real projective space* $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. It arises from $\mathbb{R}^{n+1} - \{0\}$ by identifying $x \sim y$ if $x = \lambda y$ for some $\lambda \in \mathbb{R}$.

- Describe a two-fold covering map $p \colon S^n \to \mathbb{RP}^n$ (no need to fully verify that this is a covering, as this was done in a previous exercise for $n = 2$).
- Show that $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$, when $n \geq 2$.
- Derive $\pi_1(\mathbb{RP}^n)$ for cases $n = 0$ and $n = 1$ and describe the subgroup $p_*(\pi_1(S^n))$ in these cases.

Hint: For the case $n = 1$, try to find a homeomorphism $\mathbb{RP}^1 \to S^1$ by using the degree cover $p_2 \colon S^1 \to S^1$, $p_2(z) = z^2$.

(6.4) Suppose that $p \colon \tilde{X} \to X$ is a d–fold covering map and that $\tilde{X}$ is path-connected. Prove that $\mathrm{Deck}(p)$ has at most d elements. Give examples which do and which do not realize this bound.

(6.5) As a precursor to the Borsuk-Ulam Theorem we saw that if $f \colon S^2 \to \mathbb{R}^2$ is an odd map, then there exists $x \in S^2$ such that $f(x) = 0$. Use this result to prove the following:

Given three compact sets A_1, A_2, A_3 in $\mathbb{R}^3$, show that there exists a plane that cuts each of these sets into two parts of equal measure.

Hint: Parametrize a plane H by specifying a normal vector $p \in S^2$ and a scaling parameter $t \in \mathbb{R}$, so that H is orthogonal to p and suspended at tp. You may use that for a compact set A and for every direction $p \in S^2$ there exists a parameter t such that the plane suspended at tp cuts A into two parts of equal measure, and that such a t varies continuously with p.