Example Sheet 7

Please let me (Martin) know if any of the problems are unclear or have typos. Please turn in a solution to one of Exercises (7.1) or (7.4) by 14:00 on 22/11/2018, to the dropoff box in front of the undergraduate office. If you collaborate with other students, please include their names.

For the next three problems we need the following definition. Suppose X is a topological space. We define CX to be the *cone* on X: that is,

$$CX = X \times I/(x,1) \sim (y,1) \text{ for all } x, y \in X.$$

The point $a = [(x,1)]$ is called the *apex* of the cone.

(7.1) Equip the integers $\mathbb{Z}$ with the discrete topology. Show that $C\mathbb{Z}$ is homeomorphic to the wedge sum of a countable collection of unit intervals at 1.

(7.2) Let $I_n \subset \mathbb{R}^2$ to be the line segment connecting $(0,1)$ to $(n,0)$, for $n \in \mathbb{Z}$. Set $D = \cup_{n\in\mathbb{Z}} I_n$ and equip D with the subspace topology. Show that $C\mathbb{Z}$ is not homeomorphic to D.

(7.3) For any space X, show that CX is contractible. Deduce that $\pi_1(CX, a)$ is trivial.

Hint: You may use the following fact: if X is a topological spaces, and $Y = X/\sim$ a quotient space, then

$$Y \times I \cong (X \times I)/\sim',$$

where $(x,t) \sim' (y,s)$ if $x \sim y$ and $t = s$. That is, the product topology of the quotient and the interval is the same as the quotient topology of the product (you may try to prove this!)

(7.4) Suppose G and H are nontrivial groups. Show that the free product $G * H$ is not isomorphic to $\mathbb{Z}^2$.

(7.5) Suppose that $\{G_\alpha\}$ is a countable collection of countable groups. Show that $*_\alpha G_\alpha$ is countable.

For the next two problems we need the following definition. Let $C_n \subset \mathbb{R}^2$ be the circle of radius $1/n$ centered at $(1/n, 0) \in \mathbb{R}^2$. We define $H \subset \mathbb{R}^2$, the *Hawaiian earring*, to be the union $H = \cup_{n=1}^{\infty} C_n$, equipped with the subspace topology. We take H to be a pointed space, with basepoint at $h = (0,0)$. Let $\Gamma = \pi_1(H, h)$.

(7.6) For all $n > 0$ give a retraction $r_n \colon H \to C_n$. Explain why r_n is continuous. Show that $\Gamma = \pi_1(H, h)$ is uncountable. Briefly explain why Γ is not isomorphic to

$$\pi_1\left(\vee_{n\in\mathbb{N}} S^1\right) \;\cong\; *_{n\in\mathbb{N}}\mathbb{Z}.$$