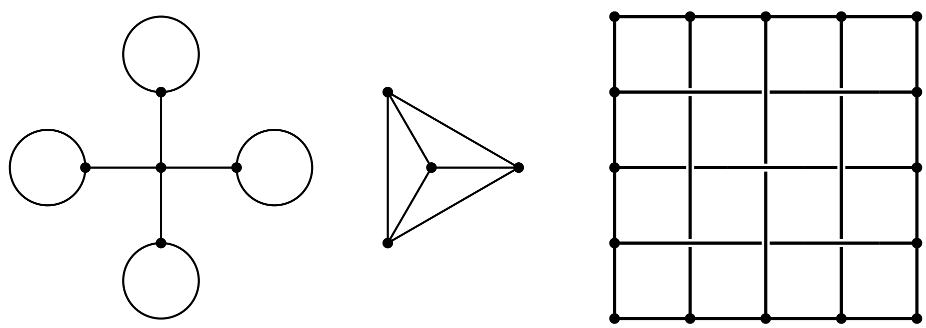
Example Sheet 8

Please let me (Martin) know if any of the problems are unclear or have typos. Please turn in a solution to one of Exercises (8.5) or (8.6) by 14:00 on 29/11/2018, to the dropoff box in front of the undergraduate office. If you collaborate with other students, please include their names.

Note: When solving exercises (8.5) or (8.6), you may assume the results of all the previous exercises.

(8.1) Show that for a finite CW complex $X = X^n$, the weak topology is the same as the quotient topology on X^n in the definition of a CW complex.



Recall that a one-dimensional CW complex is called a (topological) graph. A **subgraph** is a union of edges (1-cells) and vertices (0-cells) that is closed. A graph G is a called a **tree** if it is contractible. A vertex of a graph is called a **leaf** if it is endpoint of exactly one end of one edge. A subgraph $T \subset G$ of a graph is a maximal, or **spanning tree** if T is a tree and T contains the zero-skeleton G^0. A **cycle** is a subgraph that is homeomorphic to S^1.

(8.2) Suppose that G is a graph and $L \subset G$ is a cycle. Show that G retracts to L. Conclude that trees have no cycles.

(8.3) Suppose that T is a finite graph without cycles.

- Show that either T is a single point or T has a leaf.
- Fix a pair of distinct vertices $x, y \in T$. Show that there is a unique finite edge-path connecting x to y. This edge-path is denoted by $[x, y] \subset T$.
- Conclude that T is a tree.

(8.4) Suppose that G is a path-connected graph. Show that G contains a spanning tree.

Hint Use Zorn's Lemma (only necessary if G is not finite).

(8.5) Suppose that G is a path-connected graph and $T \subset G$ is a spanning tree where $G - T$ consists of a single edge e. Show that $\pi_1(G) \cong \mathbb{Z}$.

(8.6) Suppose that G is a path-connected graph. Show that $\pi_1(G)$ is a free product of copies of $\mathbb{Z}$. Now compute the fundamental group of each of the graphs shown in the figure above.

Hint Construct an open cover based on an open neighbourhood of a spanning tree and edges that are not in the tree.

(8.7) (Bonus) Show that connected graphs are path-connected.