Lecture 26

Proposition 26.1. *If $A \subset X$ is a subcomplex of a CW complex X, then there exists an open set $U \subset X$ with $A \subset U$, and such that U deformation retracts to A.*

An important application is that we can apply the Seifert-van Kampen Theorem to decompositions $X = A \cup B$ into subcomplexes A and B such that $A \cap B$ is again a subcomplex. For example, if $A \subset U$ and $B \subset V$, then $\pi_1(U, x_0) = \pi_1(A, x_0)$, $\pi_1(V, x_0) = \pi_1(B, x_0)$, and $\pi_1(U \cap V, x_0) = \pi_1(A \cap B, x_0)$.

In the following, we will derive an important property of the fundamental group of CW complexes, namely that it depends only on the 2-skeleton! While we can derive this as a consequence of Proposition 26.1, give outline a proof from scratch, based on the Seifert-van Kampen Theorem.

Theorem 26.2. *For a path-connected CW-complex X with $x_0 \in X^2$, the inclusion $X^2 \hookrightarrow X$ induces an isomorphism of fundamental groups $\pi_1(X^2, x_0) \cong \pi_1(X, x_0)$.*

The statement can be interpreted intuitively as saying that by studying loops, we cannot distinguish higher-dimensional topological properties. Recall, for examples, the fundamental groups of the spheres and of projective spaces:

$$\pi_1(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & \text{for } n = 1, \\ \mathbb{Z}/2\mathbb{Z} & \text{for } n \geq 2 \end{cases}, \quad \pi_1(S^n) = \begin{cases} \mathbb{Z} & \text{for } n = 1, \\ \mathbf{1} & \text{for } n \geq 2 \end{cases}$$

That is, the fundamental group does not give us more information on higher-dimensional spheres other than that any loop on it is null-homotopic. There are various ways to get higher-dimensional information. One could study higher-dimensional **homotopy groups**, arising by considering maps $S^n \to X$ instead of loops (which can be considered as maps $S^1 \to X$). A different approach is via **homology** and **cohomology**, which is the subject of more advanced courses in algebraic topology.

We begin with an observation. Note that if X is a topological space and $\varphi_\alpha \colon S^1_\alpha \to X$ is a map that attaches a 2-cell D^2_α to X, then φ_α defines a loop $f_\alpha \colon I \to X$ on X based at $\varphi_\alpha(1)$ by setting $f_\alpha(t) = \varphi_\alpha(\exp(2\pi i t))$. While this loop may not be null-homotopic in X, it is null-homotopic in

$$Y := X \sqcup D^2_\alpha / (x \sim \varphi_\alpha(x)),$$

after attaching the cell. If X is path-connected, we can choose a basepoint $x_0 \in X$ and a path $h\colon I \to X$ with $h_\alpha(0) = x_0$, $h_\alpha(1) = \varphi_\alpha(1)$, and thus get a loop $\gamma_\alpha = h_\alpha * f_\alpha * \overline{h}_\alpha$. In this way, every attaching map gives rise to a loop in Y. The inclusion $X \hookrightarrow Y$ gives rise to a map of fundamental groups $\pi_1(X, x_0) \to \pi_1(Y, y_0)$, and the class of every such loop, $[\gamma_\alpha]$, is contained in the kernel of this map.

Proposition 26.3. *Let X be a path-connected topological space and for fixed n, let $\varphi_\alpha^n\colon S_\alpha^{n-1} \to X$ be a collection of attaching maps, and set*

$$Y = X \sqcup \bigsqcup_\alpha D_\alpha^n/(x \sim \varphi_\alpha^n(x)).$$

Let $x_0 \in X$ be a point. Then

- *If $n = 2$, then*
$$\pi_1(Y, x_0) \cong \pi_1(X, x_0)/N,$$
where N is the normal subgroup generated by $[\gamma_\alpha]$, as defined above.
- *If $n > 2$, then*
$$\pi_1(Y, x_0) \cong \pi_1(X, x_0).$$

Proof. (Sketch) The proof is an application of the Seifert-van Kampen Theorem to a space $\tilde{Y}$ that deformation retracts to Y. Specifically, for each attached cell D_α^n consider a square $S_\alpha = I \times I$, a small path segment $p_\alpha\colon I \to D_\alpha^n$ with $p_\alpha(0) = h_\alpha(1) \in S_\alpha^{n-1}$ and $p_\alpha(1) \in e_\alpha^n$, and a map

$$\mu_\alpha\colon (I \times \{0\}) \cup (\{1\} \times I) \to Y, \quad (x, 0) \mapsto h_\alpha(x), \quad (1, y) \mapsto p_\alpha(y).$$

Define $\tilde{Y} = Y \sqcup \bigsqcup S_\alpha/\sim$, where the relation $\sim$ is defined by setting $x \sim \mu_\alpha(x)$ if x is in the lower-and-right boundary of S_α, and $(0, y) \sim (0, y')$ if $(0, y) \in S_\alpha$ and $(0, y') \in S_\beta$. The effect of this operation is to "lengthen" the paths from the base-point x_0 to the cells by turning them into stripes. The deformation retract of the rectangle to the lower boundary $I \times \{0\}$ induces a deformation retract of $\tilde{Y}$ to Y.

Choose points y_α in each cell e_α^n (and such that they do not lie on the path p_α). We now define the following subsets of $\tilde{Y}$:

- $A = \tilde{Y} - \bigcup_\alpha \{y_\alpha\}$;
- $B = \tilde{Y} - X$.

Since B consists of the cells e_α^n with the attached paths, it is contractible and we have $\pi_1(B, x_0) = \mathbf{1}$. By the homotopy that retracts the interior of a ball B^n without a point to its boundary, we see that $A \simeq X$. It follows that

$$\pi_1(Y, x_0) \cong \pi_1(\tilde{Y}, x_0) \cong \pi_1(X, x_0)/N,$$

where N is the normal subgroup generated by the images in $\pi_1(A, x_0)$ of elements of $\pi_1(A \cap B, x_0)$. If $n > 2$, then the cells D_α^n without a point y_α are still contractible, so

$A \cap B$ is contractible and $\pi_1(A \cap B, x_0) = \mathbf{1}$, from which the claim follows in this case. In the case $n = 2$, one gets a loop for every attached cell D^2_α that is homotopic to a loop γ_α (after a basepoint change, where the original basepoint $x_0 \in X$ is moved up the line segment to a basepoint that is in $A \cap B$). □

Proof. (of Theorem 26.2) If X is a finite-dimensional CW complex, then the statement follows from proposition 26.3 by induction: $X = X^n$ is constructed from X^{n-1} by attaching n-cells, and Proposition 26.3 tells us that this process does not alter the fundamental group if $n > 2$. If X is not finite-dimensional, we can still apply the proposition by noting that a loop γ in X is a compact subset, and therefore contained in a finite subcomplex in some X^n. Since $\pi_1(X^2, x_0) \cong \pi_1(X^n, x_0)$, every such loop is homotopic to a loop in X^2, and therefore the map $\pi_1(X^2, x_0) \to \pi_1(X, x_0)$ is surjective. To see that this is injective, let γ be a loop in X^2 that is homotopic, in X, to the constant loop via a homotopy $F\colon I \times I \to X$. As the image of F in X is compact, it is contained in a finite subcomplex X^n, and we can assume that $n > 2$. If follows that $[\gamma] = 0$ in $\pi_1(X^n, x_0)$, and we can use the injectivity of $\pi_1(X^2, x_0) \to \pi_1(X^n, x_0)$ to conclude that γ is null-homotopic in X^2. □

Note that we can get this result as a consequence of Proposition 26.1. For this, consider $X = X^n$, $A = X^{n-1}$ and $B = \bigcup_\alpha \Phi_\alpha(D^n_\alpha)$. Then $A \cap B = \bigcup_\alpha \Phi_\alpha(S^{n-1}_\alpha)$. Applying the Seifert-van Kampen Theorem to this CW decomposition, and using the fact that $\pi_1(B) = \mathbf{1}$, we get

$$\pi_1(X^n) \cong \pi_1(X^{n-1})/N,$$

with N the normal subgroup generated by loops coming from $A \cap B$. Any such loop is in $\Phi_\alpha(S^{n-1}_\alpha)$, and therefore null-homotopic if $n > 2$, but not necessarily if $n = 2$.