Proposition 26.1. If $A \subset X$ is a subcomplex of a CW complex $X$, then there exists an open set $U \subset X$ with $A \subset U$, and such that $U$ deformation retracts to $A$.

An important application is that we can apply the Seifert-van Kampen Theorem to decompositions $X = A \cup B$ into subcomplexes $A$ and $B$ such that $A \cap B$ is again a subcomplex. For example, if $A \subset U$ and $B \subset V$, then $\pi_1(U, x_0) = \pi_1(A, x_0)$, $\pi_1(V, x_0) = \pi_1(B, x_0)$, and $\pi_1(U \cap V, x_0) = \pi_1(A \cap B, x_0)$.

In the following, we will derive an important property of the fundamental group of CW complexes, namely that it depends only on the $2$-skeleton! While we can derive this as a consequence of Proposition 26.1, give outline a proof from scratch, based on the Seifert-van Kampen Theorem.

Theorem 26.2. For a path-connected CW-complex $X$ with $x_0 \in X^2$, the inclusion $X^2 \hookrightarrow X$ induces an isomorphism of fundamental groups $\pi_1(X^2, x_0) \cong \pi_1(X, x_0)$.

The statement can be interpreted intuitively as saying that by studying loops, we cannot distinguish higher-dimensional topological properties. Recall, for example, the fundamental groups of the spheres and of projective spaces:

$$\pi_1(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & \text{for } n = 1, \\ \mathbb{Z}/2\mathbb{Z} & \text{for } n \geq 2 \end{cases}, \quad \pi_1(S^n) = \begin{cases} \mathbb{Z} & \text{for } n = 1, \\ 1 & \text{for } n \geq 2 \end{cases}$$

That is, the fundamental group does not give us more information on higher-dimensional spheres other than that any loop on it is null-homotopic. There are various ways to get higher-dimensional information. One could study higher-dimensional homotopy groups, arising by considering maps $S^n \rightarrow X$ instead of loops (which can be considered as maps $S^1 \rightarrow X$). A different approach is via homology and cohomology, which is the subject of more advanced courses in algebraic topology.

We begin with an observation. Note that if $X$ is a topological space and $\varphi_\alpha : S^1_\alpha \rightarrow X$ is a map that attaches a $2$-cell $D^2_\alpha$ to $X$, then $\varphi_\alpha$ defines a loop $f_\alpha : I \rightarrow X$ on $X$ based at $\varphi_\alpha(1)$ by setting $f_\alpha(t) = \varphi_\alpha(\exp(2\pi it))$. While this loop may not be null-homotopic in $X$, it is null-homotopic in

$$Y := X \cup D^2_\alpha/(x \sim \varphi_\alpha(x)),$$
after attaching the cell. If $X$ is path-connected, we can choose a basepoint $x_0 \in X$ and a path $h : I \to X$ with $h_0(0) = x_0$, $h_0(1) = \varphi_0(1)$, and thus get a loop $\gamma_0 = h_0 * f_0 * \tau_0$. In this way, every attaching map gives rise to a loop in $Y$. The inclusion $X \hookrightarrow Y$ gives rise to a map of fundamental groups $\pi_1(X, x_0) \to \pi_1(Y, y_0)$, and the class of every such loop, $[\gamma_0]$, is contained in the kernel of this map.

**Proposition 26.3.** Let $X$ be a path-connected topological space and for fixed $n$, let $\varphi_\alpha^n : S^{n-1}_\alpha \to X$ be a collection of attaching maps, and set

$$Y = X \cup \bigcup_\alpha D^n_\alpha / (x \sim \varphi_\alpha^n(x)).$$

Let $x_0 \in X$ be a point. Then

- If $n = 2$, then
  $$\pi_1(Y, x_0) \cong \pi_1(X, x_0)/N,$$
  where $N$ is the normal subgroup generated by $[\gamma_0]$, as defined above.

- If $n > 2$, then
  $$\pi_1(Y, x_0) \cong \pi_1(X, x_0).$$

**Proof.** (Sketch) The proof is an application of the Seifert-van Kampen Theorem to a space $\tilde{Y}$ that deformation retracts to $Y$. Specifically, for each attached cell $D^n_\alpha$ consider a square $S_\alpha = I \times I$, a small path segment $p_\alpha : I \to D^n_\alpha$ with $p_\alpha(0) = h_\alpha(1) \in S^{n-1}_\alpha$ and $p_\alpha(1) \in e^n_\alpha$, and a map

$$\mu_\alpha : (I \times \{0\}) \cup (\{1\} \times I) \to Y, \quad (x, 0) \mapsto h_\alpha(x), \quad (1, y) \mapsto p_\alpha(y).$$

Define $\tilde{Y} = Y \cup \bigsqcup S_\alpha / \sim$, where the relation $\sim$ is defined by setting $x \sim \mu_\alpha(x)$ if $x$ is in the lower-and-right boundary of $S_\alpha$, and $(0, y) \sim (0, y')$ if $(0, y) \in S_\alpha$ and $(0, y') \in S_\beta$. The effect of this operation is to “lengthen” the paths from the base-point $x_0$ to the cells by turning them into stripes. The deformation retract of the rectangle to the lower boundary $I \times \{0\}$ induces a deformation retract of $\tilde{Y}$ to $Y$. Choose points $y_\alpha$ in each cell $e^n_\alpha$ (and such that they do not lie on the path $p_\alpha$).

We now define the following subsets of $\tilde{Y}$:

- $A = \tilde{Y} - \bigcup_\alpha \{y_\alpha\}$;
- $B = \tilde{Y} - X$.

Since $B$ consists of the cells $e^n_\alpha$ with the attached paths, it is contractible and we have $\pi_1(B, x_0) = 1$. By the homotopy that retracts the interior of a ball $B^n$ without a point to its boundary, we see that $A \simeq X$. It follows that

$$\pi_1(Y, x_0) \cong \pi_1(\tilde{Y}, x_0) \cong \pi_1(X, x_0)/N,$$

where $N$ is the normal subgroup generated by the images in $\pi_1(A, x_0)$ of elements of $\pi_1(A \cap B, x_0)$. If $n > 2$, then the cells $D^n_\alpha$ without a point $y_\alpha$ are still contractible, so
$A \cap B$ is contractible and $\pi_1(A \cap B, x_0) = 1$, from which the claim follows in this case. In the case $n = 2$, one gets a loop for every attached cell $D^2_\alpha$ that is homotopic to a loop $\gamma_\alpha$ (after a basepoint change, where the original basepoint $x_0 \in X$ is moved up the line segment to a basepoint that is in $A \cap B$).

**Proof.** (of Theorem 26.2) If $X$ is a finite-dimensional CW complex, then the statement follows from proposition 26.3 by induction: $X = X^n$ is constructed from $X^{n-1}$ by attaching $n$-cells, and Proposition 26.3 tells us that this process does not alter the fundamental group if $n > 2$. If $X$ is not finite-dimensional, we can still apply the proposition by noting that a loop $\gamma$ in $X$ is a compact subset, and therefore contained in a finite subcomplex in some $X^n$. Since $\pi_1(X^2, x_0) \cong \pi_1(X^n, x_0)$, every such loop is homotopic to a loop in $X^2$, and therefore the map $\pi_1(X^2, x_0) \to \pi_1(X, x_0)$ is surjective. To see that this is injective, let $\gamma$ be a loop in $X^2$ that is homotopic, in $X$, to the constant loop via a homotopy $F: I \times I \to X$. As the image of $F$ in $X$ is compact, it is contained in a finite subcomplex $X^n$, and we can assume that $n > 2$. If follows that $[\gamma] = 0$ in $\pi_1(X^n, x_0)$, and we can use the injectivity of $\pi_1(X^2, x_0) \to \pi_1(X^n, x_0)$ to conclude that $\gamma$ is null-homotopic in $X^2$.

Note that we can get this result as a consequence of Proposition 26.1. For this, consider $X = X^n$, $A = X^{n-1}$ and $B = \bigcup_\alpha \Phi_\alpha(S^n_\alpha)$. Then $A \cap B = \bigcup_\alpha \Phi_\alpha(S^n_\alpha)$. Applying the Seifert-van Kampen Theorem to this CW decomposition, and using the fact that $\pi_1(B) = 1$, we get

$$\pi_1(X^n) \cong \pi_1(X^{n-1}) / N,$$

with $N$ the normal subgroup generated by loops coming from $A \cap B$. Any such loop is in $\Phi_\alpha(S^n_\alpha)$, and therefore null-homotopic if $n > 2$, but not necessarily if $n = 2$. 
