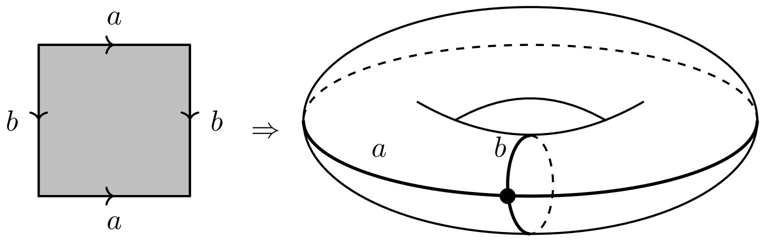
Lecture 27

27.1 Generators and relations

In this lecture we introduce a way of describing groups using generators and relations, and how to interpret these in the case of the fundamental group of a topological space. We begin by illustrating an example: the torus. Recall that the torus, $\mathbb{T}^2 = S^1 \times S^1$, has fundamental group isomorphic to $\mathbb{Z} \times \mathbb{Z}$. This was derived by noting that the fundamental group of a product is the product of fundamental groups. We now discuss a different way of describing this fundamental group, that gives more insight into the topology of the problem. The starting point is the characterization of the torus as a rectangle with opposite sides identified by gluing them together.



In this characterization, the torus is defined as

$$\mathbb{T}^2 = I \times I / \sim,$$

with $(s, 0) \sim (s, 1)$ and $(0, t) \sim (1, t)$. Let $p = (1/2, 1/2)$ be the centre of $I \times I$ and consider the open sets

$$\tilde{A} = \{x \in I \times I \colon \|x - p\| > 1/3\}$$
$$\tilde{B} = \{x \in I \times I \colon \|x - p\| < 2/3\}.$$

Let $q \colon I \times I \to \mathbb{T}^2$ be the quotient map and $A = q(\tilde{A})$, $B = q(\tilde{B})$. Thus $A \cap B$ is an annulus and $\mathbb{T}^2 = A \cup B$. We would like to derive the fundamental group of $\mathbb{T}^2$ using the Seifert-van Kampen theorem (recall that we already know this fundamental group, this is only to get a more insightful description). For this, choose a basepoint $x_0 \in A \cap B$. See Figure 27.1 for an illustration.

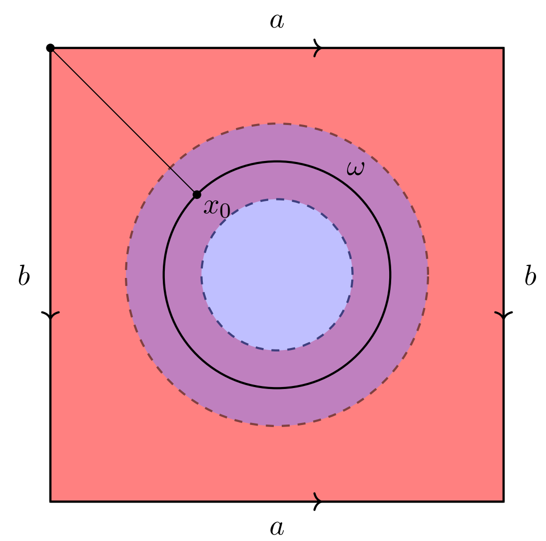
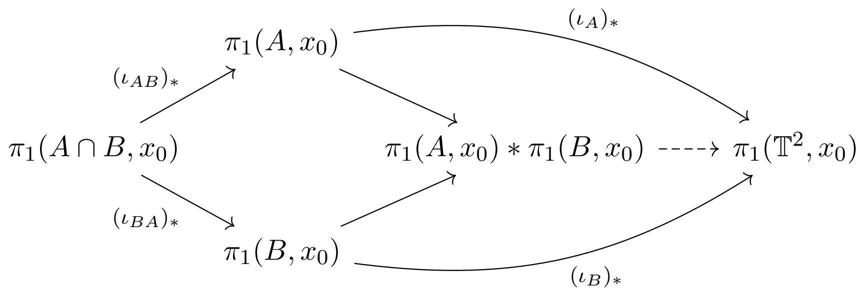


Figure 27.1: An open cover of the torus.

As usual, we denote by ι_A, ι_B the inclusions of A and B in $\mathbb{T}^2$, and by ι_{AB}, ι_{BA} the inclusions of $A \cap B$ in A and B, respectively.



The set $A \cap B$ is an annulus, that retracts to a circle. The fundamental group is generated by a loop ω at x_0. Since B is a disk, it is contractible, $\pi_1(B, x_0) = \mathbf{1}$ and

$$\pi_1(A, x_0) * \pi_1(B, x_0) = \pi_1(A, x_0).$$

Moreover,

$$(\iota_{BA})_*([\omega]) = [\iota_{BA} \circ \omega] = e_{\pi_1(B, x_0)},$$

so that the normal subgroup N of $\pi_1(A, x_0)$ is generated by

$$\iota_{AB}([\omega])_* \iota_{BA}([\omega])_*^{-1} = (\iota_{AB})_*([\omega]).$$

This is where things become interesting: what is $\pi_1(A, x_0)$, and how is $(\iota_{AB})_*([\omega])$ represented in this group? Notice that A is homotopy equivalent to a torus with a missing point in the middle (use the straight-line homotopy), and by a previous exercise this deformation retracts onto the figure-eight $S^1 \vee S^1$. Moreover, the fundamental group of this figure-eight is the free group generated by the loops a and b, that is, it consists of words in the letters $[a]$ and $[b]$. To get an explicit representation with respect to the basepoint x_0, choose a path h from x_0 to the intersection y_0 of a and b and define the loops $\gamma_a = h * a * \overline{h}$ and $\gamma_b = h * b * \overline{h}$. We then have the basepoint-change isomorphism

$$\beta_h \colon \pi_1(A, x_0) \to \pi_1(A, y_0),$$

that maps $[\gamma_a]$ to $[a]$ and $[\gamma_b]$ to $[b]$, as shown in a previous lecture. Inside A, the loop ω can now be *factored* as follows:

$$\omega \overset{\partial}{\simeq} \gamma_a * \gamma_b * \gamma_a^{-1} * \gamma_b^{-1},$$

which leads to a representation

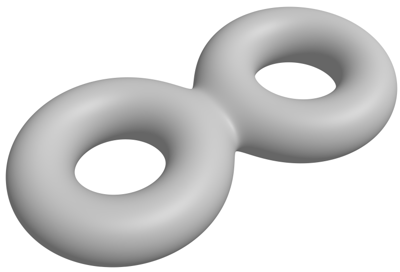
$$(\iota_{AB})_*([\omega]) = [\iota_{AB} \circ \omega] = [\gamma_a] \bullet [\gamma_b] \bullet [\gamma_a]^{-1} \bullet [\gamma_b]^{-1}.$$

If, by abuse of notation, we denote $a = [\gamma_a]$ and $b = [\gamma_b]$, then we can say that the fundamental group of $\mathbb{T}^2$ is *presented* as

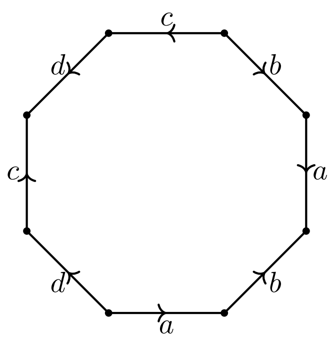
$$\pi_1(\mathbb{T}^2) = \langle a \rangle * \langle b \rangle / \langle\langle aba^{-1}b^{-1} \rangle\rangle.$$

The elements a, b are the **generators** and $aba^{-1}b^{-1}$ is a **relation**. Setting $aba^{-1}b^{-1} = 1$ amounts to requiring $ab = ba$, so that imposing this relation makes the group abelian. The resulting group is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

The torus is just a special case of a whole class of surfaces. Consider the surface S_g with g "handles". For example, the double torus:



This surface can be represented by identifying sides on an octagon:



Identifying the edges as indicated shows that the boundary is homotopic to a wedge of four spheres, $S^1 \vee S^1 \vee S^1 \vee S^1$, and in particular that all the corner points are identified with the same point. More generally, given a polygon with $4g$ edges, identifying the edges gives a boundary that is a wedge of $2g$ circles, and the resulting surface is called M_g, with g the **genus** of the surface. Using exactly the same proof as with the torus, we arrive at a fundamental group that is given by generators a_i, b_i for $1 \leq i \leq g$, and relations by the products of the elements $[a_i, b_i] := a_i b_i a_i^{-1} b_i^{-1}$, the *commutators*. A group with this structure is said to have the *presentation*

$$\langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle.$$

In general, a group G has a **presentation**

$$\langle S \mid R \rangle,$$

where S is a set of **generators** and R is a set of **relators**, is G is the free group generated by the elements of S modulo the normal subgroup generated by R,

$$G = \langle S \rangle / \langle \langle R \rangle \rangle.$$

Example 27.1. The group $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ has the presentation $\langle a \mid a^2 \rangle$.

A group is called **finitely generated** if it has a finite set S of generators, and **finitely presented**, if both S and R are finite sets.