27.1 Generators and relations

In this lecture we introduce a way of describing groups using generators and relations, and how to interpret these in the case of the fundamental group of a topological space. We begin by illustrating an example: the torus. Recall that the torus, \( T^2 = S^1 \times S^1 \), has fundamental group isomorphic to \( \mathbb{Z} \times \mathbb{Z} \). This was derived by noting that the fundamental group of a product is the product of fundamental groups. We now discuss a different way of describing this fundamental group, that gives more insight into the topology of the problem. The starting point is the characterization of the torus as a rectangle with opposite sides identified by gluing them together.

\[
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\end{array}
\Rightarrow
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\end{array}
\]

In this characterization, the torus is defined as

\[
T^2 = I \times I / \sim,
\]

with \((s, 0) \sim (s, 1)\) and \((0, t) \sim (1, t)\). Let \( p = (1/2, 1/2) \) be the centre of \( I \times I \) and consider the open sets

\[
\tilde{A} = \{ x \in I \times I : \|x - p\| > 1/3 \}
\]

\[
\tilde{B} = \{ x \in I \times I : \|x - p\| < 2/3 \}.
\]

Let \( q: I \times I \to T^2 \) be the quotient map and \( A = q(\tilde{A}), B = q(\tilde{B}) \). Thus \( A \cap B \) is an annulus and \( T^2 = A \cup B \). We would like to derive the fundamental group of \( T^2 \) using the Seifert-van Kampen theorem (recall that we already know this fundamental group, this is only to get a more insightful description). For this, choose a basepoint \( x_0 \in A \cap B \). See Figure 27.1 for an illustration.
As usual, we denote by $\iota_A, \iota_B$ the inclusions of $A$ and $B$ in $\mathbb{T}^2$, and by $\iota_{AB}, \iota_{BA}$ the inclusions of $A \cap B$ in $A$ and $B$, respectively.

The set $A \cap B$ is an annulus, that retracts to a circle. The fundamental group is generated by a loop $\omega$ at $x_0$. Since $B$ is a disk, it is contractible, $\pi_1(B, x_0) = 1$ and

$$\pi_1(A, x_0) \ast \pi_1(B, x_0) = \pi_1(A, x_0).$$

Moreover,

$$(\iota_{BA})_*([\omega]) = [\iota_{BA} \circ \omega] = e_{\pi_1(B, x_0)},$$

so that the normal subgroup $N$ of $\pi_1(A, x_0)$ is generated by

$$\iota_{AB}([\omega]) \ast \iota_{BA}([\omega])^{-1} = (\iota_{AB})_*([\omega]).$$

This is where things become interesting: what is $\pi_1(A, x_0)$, and how is $(\iota_{AB})_*([\omega])$ represented in this group? Notice that $A$ is homotopy equivalent to a torus with a missing point in the middle (use the straight-line homotopy), and by a previous
exercise this deformation retracts onto the figure-eight $S^1 \vee S^1$. Moreover, the fundamental group of this figure-eight is the free group generated by the loops $a$ and $b$, that is, it consists of words in the letters $[a]$ and $[b]$. To get an explicit representation with respect to the basepoint $x_0$, choose a path $h$ from $x_0$ to the intersection $y_0$ of $a$ and $b$ and define the loops $\gamma_a = h \ast a \ast \overline{h}$ and $\gamma_b = h \ast b \ast \overline{h}$. We then have the basepoint-change isomorphism

$$\beta_h : \pi_1(A, x_0) \to \pi_1(A, y_0),$$

that maps $[\gamma_a]$ to $[a]$ and $[\gamma_b]$ to $[b]$, as shown in a previous lecture. Inside $A$, the loop $\omega$ can now be factored as follows:

$$\partial \omega \simeq \gamma_a \ast \gamma_b \ast \gamma_a^{-1} \ast \gamma_b^{-1},$$

which leads to a representation

$$(t_{AB})_*([\omega]) = [t_{AB} \circ \omega] = [\gamma_a] \bullet [\gamma_b] \bullet [\gamma_a]^{-1} \bullet [\gamma_b]^{-1}.$$ 

If, by abuse of notation, we denote $a = [\gamma_a]$ and $b = [\gamma_b]$, then we can say that the fundamental group of $T^2$ is presented as

$$\pi_1(T^2) = \langle a \rangle \ast \langle b \rangle / \langle aba^{-1}b^{-1} \rangle.$$ 

The elements $a, b$ are the generators and $aba^{-1}b^{-1}$ is a relation. Setting $aba^{-1}b^{-1} = 1$ amounts to requiring $ab = ba$, so that imposing this relation makes the group abelian. The resulting group is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

The torus is just a special case of a whole class of surfaces. Consider the surface $S_g$ with $g$ “handles”. For example, the double torus:

This surface can be represented by identifying sides on an octagon:
Identifying the edges as indicated shows that the boundary is homotopic to a wedge of four spheres, $S^1 \vee S^1 \vee S^1 \vee S^1$, and in particular that all the corner points are identified with the same point. More generally, given a polygon with $4g$ edges, identifying the edges gives a boundary that is a wedge of $2g$ circles, and the resulting surface is called $M_g$, with $g$ the genus of the surface. Using exactly the same proof as with the torus, we arrive at a fundamental group that is given by generators $a_i, b_i$ for $1 \leq i \leq g$, and relations by the products of the elements $[a_i, b_i] := a_i b_i a_i^{-1} b_i^{-1}$, the commutators. A group with this structure is said to have the presentation

$$\langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle.$$ 

In general, a group $G$ has a presentation

$$\langle S \mid R \rangle,$$

where $S$ is a set of generators and $R$ is a set of relators, is $G$ is the free group generated by the elements of $S$ modulo the normal subgroup generated by $R$,

$$G = \langle S \rangle / \langle \langle R \rangle \rangle.$$

**Example 27.1.** The group $\mathbb{Z}_2 := \mathbb{Z} / 2\mathbb{Z}$ has the presentation $\langle a \mid a^2 \rangle$.

A group is called finitely generated if it has a finite set $S$ of generators, and finitely presented, if both $S$ and $R$ are finite sets.