Consider a graph $X = X^1$ consisting of a set of vertices $V = X^0$ and edges $(D^1_\alpha, \varphi_\alpha)$, where $\varphi_\alpha : S^0_\alpha \to X^0$ is the attaching map that assigns to each interval $D^1_\alpha$ its endpoints in the graph. Recall the characteristic map $\Phi_\alpha : D^1_\alpha \to X^1$ that maps each 1-cell to its image in the graph. We use the term edge for both a pair $(D^1_\alpha, \varphi_\alpha)$, which records combinatorial information (e.g., which are the endpoints), and for the image $\Phi_\alpha(D^1_\alpha)$ as a topological subspace of the graph.

In the following we use the convention that an edge-path in a graph is a path that can be written as a concatenation of edges:

$$\gamma = e_1 \ast \cdots \ast e_m,$$

where each $e_i$ is an edge (in the subspace-sense). Similarly, an edge loop is an edge-path that ends where it starts.

28.1 From CW complexes to groups and back

Given a CW complex $X$ and $x_0 \in X$, we can compute a presentation of the fundamental group $\pi_1(X, x_0)$. As the path-components that do not contain $x_0$ do not enter the fundamental group, we may replace $X$ with the path component containing $x_0$. In addition, we can move the basepoint to lie in $X^1$ (or even $X^0$), as this does not change the structure of the fundamental group. Finally, we can restrict to the 2-skeleton, and hence assume without lack of generality that $X = X^2$ is a path-connected, two-dimensional CW complex. To compute the fundamental group we proceed as follows:

1. Find a spanning tree of $T \subset X^1$. This can be done, for example, using Dijkstra’s algorithm. Let $A$ be the set (not union!) of edges that are not in the tree. Pasting such an edge to the graph $T$ gives a subgraph that is homotopic to a circle $S^1$, i.e., an edge-cycle. As shown in the exercises, we can describe the fundamental group of $X^1$ as generated by these edge-cycles.

$$\pi_1(X^1, x_0) \cong \ast_{e \in A} \mathbb{Z}.$$
Every edge not in $T$ gives a loop when adding it to $T$, and conversely every loop in $X^1$ based at $x_0$ is homotopic to a combination of such edge-cycles (loops that consist of traversing a cycle that arises by adding $a \in A$ along edges).

2. Let $e^2_\alpha \subset X^2$ (here we identify the open 2-cells with their images in $X^2$) be a 2-cell and 
\[
\varphi_\alpha : S^1_\alpha \to X^1
\]
the attaching map. Recall that $\gamma_\alpha(t) = \varphi_\alpha(\exp(2\pi it))$ is a loop, and hence homotopic to an edge-loop (a loop consisting of edges). Let $x_1 \in \varphi_\alpha(S^1_\alpha)$ and let $g_\alpha : I \to X^1$ be a path with $g_\alpha(0) = x_0$ and $g_\alpha(1) = x_1$. Then
\[
\omega_\alpha = [g_\alpha * \gamma_\alpha * g_\alpha] \in \pi_1(X^1, x_0)
\]
and therefore corresponds to a reduced word $u_\alpha$ in $A$. Set $U = \{u_\alpha\}_\alpha$.

We claim that
\[
\pi_1(X, x_0) \cong \pi_1(X^1, x_0)/\langle \langle U \rangle \rangle,
\]
or in other words, that the fundamental group of $X$ with base $x_0$ is presented as $\langle A \rangle U$. In fact,

- The union of the cells $e^2_\alpha$ together with the paths joining them to $x_0$ form a contractible subcomplex: $\pi_1(A, x_0) \cong 1$.

- Choose points $y_\alpha \in e^2_\alpha$ inside each of the cells $e^2_\alpha$ and define the subset $B = X^2 - \bigcup_\alpha \{y_\alpha\}$. Then $B$ retracts to $X^1$ (we poke a “hole” into each of the 2-cells attached to $X^1$), and $\pi_1(B, x_0) \cong \pi_1(X^1, x_0)$.

- We have $X^2 = A \cup B$ and $A \cap B$ consists of precisely those edge-cycles starting at $x_0$ that make up loops homotopic to the boundaries of 2-cells, or in other words, the images of $S^1_\alpha$ under the attaching maps. Therefore, each element of $\pi_1(A \cap B, x_0)$ represents a word in $U$.

- The fundamental group of $X$ is therefore given as
\[
\pi_1(X, x_0) \cong \pi_1(X^2, x_0) \cong \pi_1(A, x_0) * \pi_1(B, x_0)/\langle \langle U \rangle \rangle \cong \pi_1(X^1, x_0)/\langle \langle U \rangle \rangle.
\]

**Figure 28.1:** The graph $X^2$
The construction is best visualized as in Figures 28.1 and 28.2. In summary:

- Every cycle in the underlying graph $X^1$ corresponds to a loop based at $x_0$ that moves along edges from $x_0$ to the cycle, around the cycle, and back to $x_0$. Every such cycle corresponds to a generator of the fundamental group $\pi_1(X^2, x_0)$;

- Every loop in $X^2$ can be represented as a combination of such cycles-paths along edges. This corresponds to a reduced word in the generators of $\pi_1(X^2, x_0)$;

- A loop is null-homotopic if it is homotopic to the boundary of a 2-cell in $X^2$. Such loops corresponds to a relation on the set of words in $\pi_1(X^2, x_0)$.

The Seifert-van Kampen Theorem merely provides a means to formalizing the above intuitive procedure.

**Example 28.1.** Recall the characterization of real projective space as CW complex. Recall the cell decomposition of $\mathbb{R}P^2$ into one 0-cell, one 1-cell and one 2-cell, which can be visualized as follows.
Even though we see two points and two arcs labelled with $\gamma$, the points are identified to make one point, and the lines are identified (glued together) along the direction of the arrow. The 1-skeleton $X^1$ of this is just a loop consisting of a single edge, and a spanning tree consists of the only vertex in this graph. The generator of the fundamental group is thus this one cycle, whose class we denote by $a$ (say). For the relation, we look at the loop that bounds the 2-cell: as seen in the image, this loop consists of going around the cycle twice, so it is represented by $a^2$. Therefore, the fundamental group is presented by $\langle a \mid a^2 \rangle$, and the corresponding group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

**Example 28.2.** Just as there are different ways of describing a topological space as a CW complex, there are different ways to “present” the fundamental group. We discuss this using an illustrative example, the **Klein bottle** $K$.

The image shows an attempted embedding of the Klein bottle into $\mathbb{R}^3$; this is not possible without self-intersections. As a CW complex, the Klein bottle is usually described like a Möbius strip, but with the top and bottom sides identified as well.
The underlying 1-skeleton \( X^1 \) consists of two loops, \( a \) and \( b \), while there is only one vertex (by following the identification of the boundaries of the rectangle as indicated by the arrows, one sees that all the corners are collapsed to a single point). Therefore, the generators are the classes corresponding to the cycles \( a \) and \( b \) (which we will also denote by \( a \) and \( b \)). The single relation is the loop that forms the boundary of the rectangle and is given by \( baba^{-1} \) (formally, the class in the fundamental group of \( X^1 \) that is generated by the loop \( b * a * b * a^{-1} \)). We therefore get a presentation 

\[
\langle a, b \mid baba^{-1} \rangle.
\]

In other words, all the elements in this group are words in \( a \) and \( b \) (or equivalently, binary sequences), where every occurrence of \( baba^{-1} \) is replaced with the empty word.

One might, of course, ask whether this group looks like a more familiar group, or whether it can be described in a simpler way. One way to arrive at such a simpler representation is to use a different CW-complex representation.

In this case, we can add an additional cycle \( c \) and then remove the cycle \( b \). The resulting picture can then be visualized as follows.

The resulting group presentation is then 

\[
\langle a, c \mid a^2 c^2 \rangle.
\]
This is somehow easier to interpret.

By now we should have an idea of how to get a group out of a CW complex. Conversely, any group presentation leads to a topological space (in fact, a surface) whose fundamental group is isomorphic to the given group.

**Theorem 28.3.** For every group \( G \) there exists a path-connected two-dimensional CW complex \( X_G \) such that

\[
\pi_1(X_G) \cong G.
\]

**Proof.** Consider a presentation of the group (generators and relators). Construct the one skeleton \( X^1 \) of \( X_G \) as a wedge (one point union) of circles \( S^1 \), with one circle per generator. Every relator describes a loop in \( X^1 \): for example if \( ab^{-1}c^2 \) is a relator, then the loop is given by going around \( a \) once, around \( b \) once in the opposite direction, and then twice around \( c \). For each such relator take a 2-cell \( D^2_\alpha \) with boundary \( S^1_\alpha \) and define an attaching map

\[
\varphi_\alpha : S^1_\alpha \to X^1
\]

that maps the circle onto the loop specified by the relators. The resulting CW-complex \( X = X^2 \) is then a two-dimensional CW complex whose fundamental group is, by construction, isomorphic to \( G \). \( \square \)